Queen Mary
University of London

## B.Sc. Examination by course unit 2014

## MTH6120 Further Topics in Mathematical Finance (Exam for 2012/13 and 2013/14 cohort)

**Duration: 2 hours**

**Date and time: 27.05.2014, 10:00am**

---

**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt all questions. Marks awarded are shown next to the questions.**

**Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.**

**Complete all rough workings in the answer book and cross through any work which is not to be assessed.**

**Important note: the Academic Regulations state that possession of unauthorized material at any time by a student who is under examination conditions is an assessment offence and can lead to expulsion from QML.**

**Please check now to ensure you do not have any notes, mobile phones or unauthorized electronic devices on your person. If you have any, then please raise your hand and give them to an invigilator immediately. Please be aware that if you are found to have hidden unauthorized material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorized material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. Disruption caused by mobile phones is also an examination offence.**

**Exam papers must not be removed from the examination room.**

**Examiner:** Dr. A. Baule

---

 **TURN OVER**

## Question 1

(a) Explain what the word "arbitrage" means. Why is the concept of arbitrage important in the context of mathematical finance?

**[4 marks]**

(b) Write down the definition of geometric Brownian motion. What is the condition for risk-neutral geometric Brownian motion?

**[6 marks]**

(c) Assume that $X$ is a Gaussian random variable with mean $0$ and variance $\sigma^2$. Use the rule for the transformation of probability density functions to show that $X$ can be expressed as

$$X = \sigma Z,$$

where $Z$ is a Gaussian random variable with mean $0$ and variance $1$.

**[5 marks]**

(d) Consider a financial contract, which has the random payoff $g(S(t_2))$ at the time $t_2$, where $g : \mathbb{R} \to \mathbb{R}$ is a function and $S(t)$ a risk-neutral stochastic process. Give an expression for the price of the financial contract at a time $t_1 < t_2$ under the condition of no-arbitrage and continuous compounding with a constant nominal interest rate $r$. Give reasons for your expression.

**[5 marks]**

(e) Consider a European $(K, t)$ put option whose return at expiration time $t$ is capped by the amount $B > 0$. That is, the payoff at time $t$ is given by

$$\min((K - S(t))^+, B).$$

Show that the price today of such an option can be expressed in terms of today's prices of two plain (uncapped) options.

**[5 marks]**

## Question 2

(a) Suppose you know the price of a European put option with a certain strike price and expiration time. You also know the current price of the underlying share and the nominal interest rate. How can you determine the price of a European call option with the same strike price and expiration time?

**[4 marks]**

(b) Explain the difference between a European call option and an American call option. Does the difference matter in practice?

**[3 marks]**

(c) State and prove the generalized law of one price.

**[4 marks]**

(d) Consider a put option with strike price $K$ and expiration time $t$ on an index $I(t) = \sum_{j=1}^{n} w_j S_j(t)$. Show that the price $P_I(K,t)$ of this index put option is always smaller than or equal to the weighted sum of put options on the individual shares $P_j(K_j,t)$:

$$P_I(K,t) \leq \sum_{j=1}^{n} w_j P_j(K_j,t),$$

where the strike price $K$ is given by $K = \sum_{j=1}^{n} w_j K_j$.

**[6 marks]**

(e) Consider the random variable

$$F(t) = \prod_{i=1}^{N(t)} Y_i,$$

where $N(t)$ is a Poisson process with rate parameter $\lambda$ and the $Y_i$ are identically and independently distributed Gaussian random variables with mean $\mu$ and variance $\sigma^2$. Calculate the second moment $\langle F^2(t) \rangle$.

**[8 marks]**

## Question 3

(a) Consider the function $u(x) = \log(1 + bx^2)$ for $x \geq 0$ and $b > 0$. Is $u(x)$ suitable as a utility function? Give reasons.

**[4 marks]**

(b) Consider the utility function $u(x) = 1 - e^{-bx}$ with $b > 0$. Show that maximizing the expected utility $\langle u(W) \rangle$ for a Gaussian random variable $W$ is equivalent to maximizing

$$b \langle W \rangle - \frac{1}{2} b^2 \mathrm{Var}(W).$$

**[5 marks]**

(c) Four different investments have the same expected values of the rate of return but different variances $v_j^2 = \frac{1}{j^2}$, where $j = 1, 2, 3, 4$. The investments are uncorrelated with each other. Construct the optimal portfolio if you have a total capital of $w$ to invest.

**[5 marks]**

(d) Using mathematical notation, explain what the knapsack optimisation problem is. (You do not need to discuss how to solve it.)

**[4 marks]**

(e) Find the optimum investment $(x_1, x_2)$ if a total amount $x_1 + x_2 = 10$ is to be invested between two projects having return functions $f_1(x) = \log(x + 3)$ and $f_2(x) = 2\sqrt{x}$, assuming that $x_1$ and $x_2$ take positive values which are not necessarily integers. Prove that you have indeed found a maximum in the total return.

**[7 marks]**

## Question 4

(a) Explain the payment structure of a bond. Write down an equation for the price of a bond with face value $F$ and annual coupon payments $C$, given the yield-to-maturity $r$.

**[5 marks]**

(b) Explain what the spot rate curve is. Explain how you can determine the spot rate curve from the prices of unit zero coupon bonds with different maturities.

**[5 marks]**

(c) Consider a series of monthly payments $\{C_1, C_2, ...\}$, made at the end of each month over $n$ years. Assume a constant nominal interest rate $r$ and continuous compounding. Calculate the present value, the effective duration and the convexity of the cash flows. How does the present value change when the interest rate $r$ increases?

**[10 marks]**

(d) You invest $£100$ into a bank account with variable interest rates. Assume that the interest rate in the $i$th year is given by a random variable $R_i$, which can assume one of the values $\{r_1, r_2, r_3\}$ with probabilities $P(R_i = r_j) = p_j$ for $j = 1, 2, 3$. What is the random value $S_n$ of your investment after $n$ years? Calculate the expected value of $S_n$.

**[5 marks]**

---

**End of Paper**