Queen Mary University of London

Main Examination period 2018

MTH6109 / MTH6109P: Combinatorics

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

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Exam papers must not be removed from the examination room.

Examiners: J. Ward, M. Jerrum

For questions 1 and 2: You do not need to simplify any binomial coefficients, factorials, or large powers. Additionally, you may (but are not required to) provide a brief justification for how you arrived at your answer. This may be counted toward partial marks even if your final answer is incorrect.

Question 1. [18 marks] There is an athletics club that has 12 members.

(a) The club enters a 4 person team to run a relay race. To do this, they must produce an ordered list of 4 members. How many different such lists can be chosen? [**4**]

(b) The club selects a set of 5 members to compete in a regional contest. How many different ways can the set of 5 members be chosen? [**4**]

(c) The members of the club split up into teams of size 2 to play tennis. How many different ways can the 12 members of the club be divided into 6 disjoint sets of size 2? [**5**]

(d) The club buys a box of 20 identical protein bars. How many different ways can they divide the 20 bars amongst all the members, if each member must receive at least one bar? [**5**]

Question 2. [17 marks]

(a) State the **General Inclusion-Exclusion Principle**. [**6**]

Now, consider the following 3 disjoint sets of symbols:

- D is the set of all 10 digits, $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- U is the set containing all 26 upper-case letters.
- S is the following set of 4 special symbols: $\{\$, \%, *, !\}$.

Let $A = D \cup U \cup S$ and consider the set of all sequences of length 10 over A. Answer each of the following questions about this set of sequences.

(b) How many such sequences do **not** contain a symbol from D? [**3**]

(c) How many such sequences do **not** contain a symbol from either D or U? [**3**]

(d) How many such sequences contain **at least one** symbol from **each** of D, U, and S? [**5**]

Question 3. [18 marks]

(a) Solve the following recurrence with given initial conditions. That is, give a closed-form solution for a_n:

$$a_n = a_{n-1} + 6a_{n-2}, \quad \text{for all } n \geq 2$$
$$a_0 = 5, \quad a_1 = 10$$

[**9**]

(b) Let

$$R(x) = \frac{7}{1+3x} + \frac{3}{1-4x}$$

be the generating series for the sequence $r_0, r_1, r_2, \ldots$. Give a closed-form expression for r_n. [**9**]

Question 4. [18 marks]

(a) Give 3 different characterisations of trees. That is, give 3 different statements about a graph G, each of which is true if and only if G is a tree. [**9**]

(b) State Hall's Condition, which holds if and only if a bipartite graph $G(V, E)$ with sides L and R (where $V = L \cup R$) has a matching saturating L. [**4**]

(c) You are scheduling a set of job interviews. There are 6 candidates: Alice, Bob, Cynthia, Dmitiri, Erica, and Faiz. There are 6 time slots available, starting at: 1PM, 2PM, 3PM, 4PM, 5PM, and 6PM. The following table shows which candidates are available for which times: each of the time slots for which a candidate is available is marked with an X.

	1PM	2PM	3PM	4PM	5PM	6PM
Alice	X	X	X			
Bob	X		X			
Cynthia	X	X				
Dmitri		X	X			
Erica				X	X	X
Faiz		X		X		X

Your task is to assign each candidate an interview time so that: (1) candidates are only interviewed at times when they are available, and (2) no two candidates are interviewed at the same time. Is it possible to find such an assignment? If so, give the assignment. If not, say why. [**5**]

Question 5. [19 marks]

(a) State Euler's formula, which holds for any planar embedding of a connected graph. [**4**]

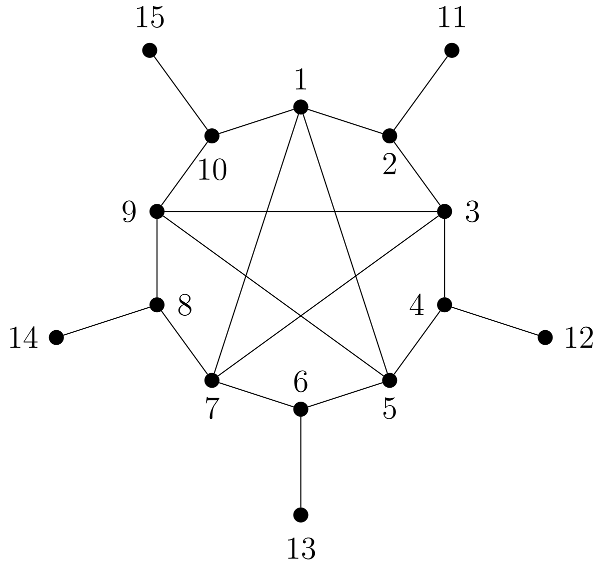
(b) Let $G(V, E)$ be a graph and recall that for every $v \in V$, we define $\deg(v)$ as the number of edges incident on v. Show that:

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)| \ .$$ [**4**]

(c) What is a **subdivision** of a graph G? [**3**]

(d) State Kuratowski's Theorem characterising planar graphs (you do not need to define any well-known graphs appearing in the statement of the theorem). [**4**]

(e) Is the following graph planar? If so, give (that is, draw) a planar embedding for it. If not, say why. [**4**]



Question 6. [10 marks]

(a) Are the following Latin squares of order 4 orthogonal? Justify your answer.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

[**4**]

(b) Prove that for any $n \geq 1$, there are at most $(n - 1)$ mutually orthogonal Latin squares of order n. [**6**]

End of Paper.