Queen Mary University of London

Main Examination period 2022 – May/June – Semester B

MTH6108: Coding Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: I. Morris, A. Saha

Question 1 [13 marks].

(a) Consider the code $\mathcal{C}$ over the alphabet $\mathbb{A} = \{\text{A, B, } \ldots \text{, Y, Z}\}$ given by:

$$\mathcal{C} := \{\text{BARNET, CAMDEN, EUSTON, BALHAM, MORDEN}\}.$$

Compute the minimum distance of $\mathcal{C}$. What is the largest integer $t \geq 0$ such that $\mathcal{C}$ is t-error-detecting?

Include an appropriate justification for your answer to both parts of the question. [**7**]

(b) Let $\mathcal{D}$ be a code over an alphabet $\mathbb{A}$. Suppose that $\mathcal{D}$ is 2-error-correcting but not 3-error-correcting.

(i) What are the possible values that $d(\mathcal{D})$ might take? [**3**]

(ii) For which $t \geq 0$ is it definitely true that $\mathcal{D}$ is t-error-detecting? [**1**]

(iii) For which $t \geq 0$ is it definitely **not** true that $\mathcal{D}$ is t-error-detecting? [**1**]

(iv) For which $t \geq 0$ is it **possibly** true – but not certain – that $\mathcal{D}$ is t-error-detecting? [**1**]

Explain your answers with reference to results from the course.

Question 2 [16 marks]. Decide which of the following statements is true and which is false. Justify your answer in each case, stating which results from the course you use in your answer (if any).

(a) $A_2(13, 9) = 2$. [**5**]

(b) $A_3(8, 5) > 50$. [**5**]

(c) $A_4(5, 3) = 64$. [**6**]

Question 3 [10 marks].

Recall that the **rate** of a q-ary (n, M, d)-code $\mathcal{C}$ is defined to be the quantity

$$R(\mathcal{C}) = \frac{\log M}{n \log q}.$$

(a) If $\mathcal{E}$ is a linear $[n, k]$-code over $\mathbb{F}_q$, what is the rate of $\mathcal{E}$? [**3**]

(b) Calculate the rate of the Reed-Muller code $\mathcal{R}(2, 5)$. [**3**]

(c) Calculate the rate of the Hamming code $\mathrm{Ham}(4, 4)$. [**4**]

Justify your answer in each case.

Question 4 [9 marks]. Let $\mathcal{C} = \{0000, 0101, 1100, 1011, 0110\}$ be a code over the alphabet $\mathbb{A} = \{0, 1\}$. Construct a nearest-neighbour decoding process $f\colon \mathbb{A}^4 \to \mathcal{C}$ and use it to calculate the word error probability of the word 1100, assuming that the symbol error probability is $p = \frac{1}{5}$. **[9]**

Question 5 [14 marks].

(a) Consider the linear code over $\mathbb{F}_2$ with generator matrix

$$G = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

(i) List all of the words in the code $\mathcal{C}$. **[4]**

(ii) Write down the minimum distance of $\mathcal{C}$. **[2]**

(iii) Construct a Slepian array for $\mathcal{C}$ and use it to decode the word 11110. **[8]**

Question 6 [18 marks].

(a) Consider the linear code $\mathcal{C}$ over $\mathbb{F}_2$ with parity-check matrix given by

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

(i) Construct a syndrome lookup table for $\mathcal{C}$. Your answer should include any calculations involved in the construction. **[8]**

(ii) Use your syndrome lookup table to decode the word 101010. **[3]**

(b) (i) Let $\mathcal{D} \subseteq \mathbb{F}_q^n$ be a linear code of dimension k, where q is a prime power. Define the **dual code** $\mathcal{D}^\perp$. Show that if $\mathcal{D} = \mathcal{D}^\perp$ then n must be even. Indicate which results from the course you use in your answer (if any). **[4]**

(ii) Show that the code $\mathcal{C}$ from part (a) is not equal to its own dual $\mathcal{C}^\perp$. **[3]**

Question 7 [9 marks].
Consider the following three matrices with entries in the field $\mathbb{F}_7$:

$$H_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 4 & 0 & 2 & 1 & 3 & 1 & 6 & 5 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$H_3 = \begin{pmatrix} 2 & 0 & 4 & 2 & 5 & 1 & 1 & 2 \\ 0 & 3 & 1 & 4 & 6 & 3 & 4 & 1 \end{pmatrix}.$$

For each of the three matrices H_1, H_2, H_3 decide whether or not that matrix is a parity-check matrix for (a version of) the Hamming code $\mathrm{Ham}(2, 7)$. Justify your answer for each matrix. [**9**]

Question 8 [11 marks].

(a) Let $\mathcal{C}$ be the linear code over $\mathbb{F}_7$ with generator matrix

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

Write down a parity-check matrix for the code $\mathcal{C}$ and use it to calculate the minimum distance of $\mathcal{C}$. Explain whether or not $\mathcal{C}$ is an MDS code. [**6**]

(b) Let $\mathcal{D}$ be the linear code over $\mathbb{F}_5$ whose generator matrix is also given by

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 3 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

(Note that this matrix looks the same as the matrix as in part (a) above, but its entries now belong to a different field $\mathbb{F}_5$.) Write down a parity-check matrix for the code $\mathcal{D}$ and use it to decide whether or not $\mathcal{D}$ is an MDS code. [**5**]

End of Paper.