Main Examination period 2020 – January – Semester A

MTH6107/MTH6107P: Chaos and Fractals

Duration: 2 hours

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You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

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Examiners: A. Clark, O. Jenkinson

Question 1 [25 marks].

(a) Consider a function $f : \mathbb{R} \to \mathbb{R}$.

(i) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **fixed point** for f? **[1]**

(ii) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **periodic point** for f? **[1]**

(iii) How is the **prime period** of a periodic point defined? **[1]**

(iv) What does it mean to say that a point $x_0 \in \mathbb{R}$ is a **pre–periodic point of period n** for f? **[1]**

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = -x^2 + 10x - 20$.

(i) Compute the values of all fixed points of f. **[4]**

(ii) For each fixed point determine whether the fixed point is attracting, repelling or neither. **[4]**

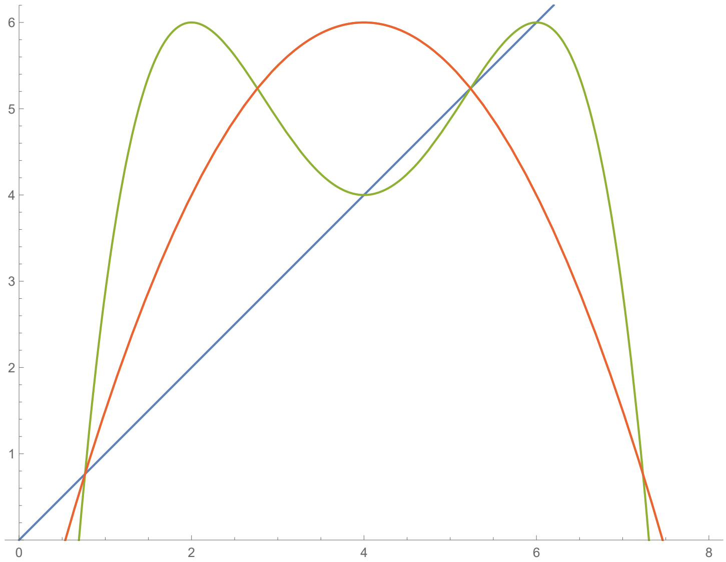
(iii) Find a pre–periodic point that is not periodic, or give a reason why there is none. **[2]**

(iv) Give the basin of attraction of each attracting fixed point. **[5]**

(c) If $f : \mathbb{R} \to \mathbb{R}$ is a C^1–diffeomorphism with $f'(x) < 0$ for all x, show that f must have a fixed point. **[6]**

Question 2 [25 marks].

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = -x^2/2 + 4x - 2$.



Graph of $y = f(x)$, $y = f^2(x)$ and $y = x$.

(i) Find all points of prime period 2 for f and verify that the points have prime period 2. **[4]**

(ii) For each 2–cycle, determine whether the 2–cycle is attracting, repelling or neither. **[4]**

(iii) Using the graph or otherwise, find the basin of attraction of any attracting 2–cycle. (For this, you may find it useful to let p denote the smaller fixed point, let p' denote the larger fixed point, and let q denote the unique point $q \neq p$ satisfying $f(q) = p$; there is no need to calculate the values of p, p' or q.) **[5]**

(iv) What prime periods occur for f? **[2]**

(b) State **Sharkovskii's Theorem** concerning the existence of periodic points of specified prime periods for a continuous map f from $\mathbb{R}$ to itself. **[5]**

(c) State whether each of the following statements is true or false.

(i) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with a point of prime period 7, but no point of prime period 5. **[1]**

(ii) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with exactly one point of prime period 3, and 5 points of prime period 5. **[1]**

(iii) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with a point of prime period 12 but no point of prime period 14. **[1]**

(iv) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with a point of prime period 8, and no point of prime period 4. **[1]**

(v) There is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with no periodic points. **[1]**

Question 3 [25 marks].

(a) Let $f_\mu : \mathbb{R} \to \mathbb{R}$ denote the logistic map, defined by $f_\mu(x) = \mu x(1-x)$, for real positive values of the parameter μ.

(i) Define briefly the **period-doubling bifurcation cascade**. **[4]**

(ii) Define the **Feigenbaum constant**. **[3]**

(b) For $a > 0$ define the family of maps $g_a : \mathbb{R} \to \mathbb{R}$ by $g_a(x) = 1 - ax^2$. Solving the equation $g_a^2(x) = x$ algebraically yields the four solutions:

$$x = \frac{-1+\sqrt{1+4a}}{2a}, \frac{-1-\sqrt{1+4a}}{2a}, \frac{1+\sqrt{4a-3}}{2a}, \frac{1-\sqrt{4a-3}}{2a}.$$

(i) Identify the fixed points of g_a. **[4]**

(ii) Show that g_a has an attracting fixed point for $0 < a < 3/4$. **[4]**

(iii) For which values of a does g_a have a cycle of prime period 2? **[4]**

(iv) For $\frac{3}{4} < a < \frac{5}{4}$ show that g_a has an attracting 2–cycle. **[4]**

(v) Based on part (iv), what is the smallest value of $a > 0$ for which g_a could have a point of prime period 4? (You do not need to determine whether g_a has a point of prime period 4 for this value of a.) **[2]**

Question 4 [25 marks].

(a) Show that

$$h : [0,1] \to [-1,1]; \;\; h(x) = 2x - 1$$

provides a topological conjugacy from

$$f : [0,1] \to [0,1]; \;\; f(x) = 4x(1-x)$$

to

$$g : [-1,1] \to [-1,1]; \;\; g(x) = 1 - 2x^2.$$

[4]

(b) Let

$$S_2 = \{(s_1, s_2, ...) \; : \; s_i \in \{0,1\} \text{ for all } i\}$$

with shift map $\sigma_2 : S_2 \to S_2$; $\sigma_2\left((s_1, s_2, s_3, ...)\right) = (s_2, s_3, ...)$, and let

$$S_4 = \{(s_1, s_2, ...) \; : \; s_i \in \{0,1,2,3\} \text{ for all } i\}$$

with shift map $\sigma_4 : S_4 \to S_4$; $\sigma_4\left((s_1, s_2, s_3, ...)\right) = (s_2, s_3, ...)$.

(i) How many points of prime period 2 does σ_2 have and how many points of prime period 2 does σ_4 have? **[4]**

(ii) Are σ_2 and σ_4 topologically conjugate? Justify your answer. **[4]**

(c) Define the **Lyapunov exponent** of a differentiable map $f : \mathbb{R} \to \mathbb{R}$ at a point $x_0 \in \mathbb{R}$, and for the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x^2 + 2x$ compute the Lyapunov exponent at the point $x_0 = 0$. **[5]**

(d) (i) State Devaney's definition of **chaos**. **[4]**

(ii) For each of the following functions, determine whether the function satisfies Devaney's definition of chaos.

A. $f : [0,1] \to [0,1]\,; f(x) = 4x(1-x)$ **[1]**

B. $f : [0,1] \to [0,1]\,; f(x) = x(1-x)$ **[1]**

C. $f : [0,1] \to [0,1]\,; f(x) = \begin{cases} 2x & \text{for } x \leq 1/2 \\ 2 - 2x & \text{for } x \geq 1/2 \end{cases}$ **[1]**

D. $f : [0,1] \to [0,1]; f(x) = x^4$. **[1]**

Question 5 [25 marks].

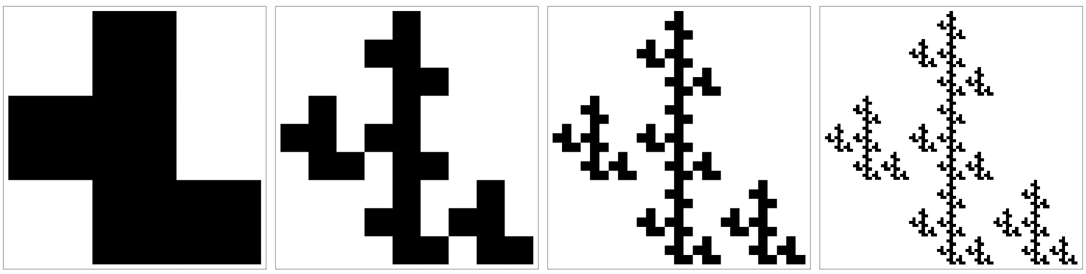
(a) Define the **box-counting dimension** of a bounded subset of $\mathbb{R}^n$, assuming it exists. **[4]**

(b) Consider the iterated function system given by the two functions $\{f_1, f_2\}$ $f_i : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x/5$ and $f_2(x) = (x+4)/5$. Let $C_1 = f_1([0,1]) \cup f_2([0,1])$ and let $C_2 = f_1 \circ f_1([0,1]) \cup f_2 \circ f_1([0,1]) \cup f_2 \circ f_1([0,1]) \cup f_2 \circ f_2([0,1])$.

(i) Calculate the endpoints of the intervals forming C_1 and C_2. **[4]**

(ii) If we continue in this fashion forming C_n for $n = 1, 2, \ldots$, then $C = \cap_{n=1}^{\infty} C_n$ is the unique limit set of the IFS $\{f_1, f_2\}$. Find the box–counting dimension of this limit set C. (You may assume the box–counting dimension exists.) **[5]**

(c) For this problem we will consider a set S in $\mathbb{R}^2$ constructed via a recursive process similar to the Sierpinski Carpet. We begin in the initial Step 0 with the unit square $[0,1] \times [0,1]$. In Step 1 we subdivide the original square into 9 equal squares forming 3 rows of 3 squares and we then remove the first and final square in the first row of squares (going left to right), we remove the final square of row 2, and we remove the first square of row 3. In general, going from Step n to Step $n+1$ we subdivide each of the remaining squares into 9 equal squares and remove the first and final square in the first row of squares, we remove the final square of row 2, and we remove the first square of row 3. S is the set that is formed by the intersection of all sets from each stage. Below we picture Step 1 through Step 4 in the construction of S.



(i) Find the (Lebesgue) measure of S. **[4]**

(ii) Find the box–counting dimension of S. (You may assume it exists.) **[6]**

(iii) Is S a fractal? Justify your answer. **[2]**

End of Paper.