Queen Mary
University of London

Main Examination period 2018

MTH5117: Mathematical Writing

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: I. Tomašić and B. Noohi

Marks are deducted for incorrect grammar/spelling. In a question, or part of a question, the notation **[**$\not\in$, n**]** indicates that the answer should not contain any mathematical symbols whatsoever, apart from numerals. The integer n—when present—prescribes the **approximate** length (in words). In the absence of this notation, mathematical symbols may be used freely.

Question 1. [25 marks]

For each of the following mathematical objects provide two levels of description:

(i) a coarse description, which only identifies the class to which an object belongs (set, function, etc.) **[**$\not\in$**]**;

(ii) a finer description, which describes the object in question as accurately as possible **[**$\not\in$**]**.

(a) $(\sqrt{2} \notin \mathbb{Q}) \wedge (2 + 3 = 5)$. **[5]**

(b) $\exp^{-1}(1)$. **[5]**

(c) $\sin^{-1}(\{0\})$. **[5]**

(d) $x^2 + y^2 = 1$. **[5]**

(e) $\{ (x, y) \in \mathbb{Q}^2 : x^2 + y^2 = 1 \}$. **[5]**

Question 2. [25 marks]

(a) Express each of the following statements with symbols, **using at least one quantifier**.

(i) The function $\sin : \mathbb{R} \to [-1, 1]$ is surjective. **[4]**

(ii) The function $\exp : \mathbb{R} \to \mathbb{R}$ is bounded. **[4]**

(iii) The equation $x^3 - 1 = 0$ has at least two distinct real solutions. **[4]**

(iv) There is no largest real number. **[4]**

(v) The number $2^7 - 1$ is prime. **[4]**

(b) For each statement above, state whether:

- it is definitely TRUE,
- it is definitely FALSE, or
- it is UNKNOWN, i.e., there is not enough information to determine whether it is true or false.

[5]

Question 3. [18 marks]

Consider the implication:

For every integer n, if n is odd, then n^2 is odd.

(a) Write its (i) contrapositive, (ii) converse, (iii) negation. **[9]**

(b) For each of (i), (ii), (iii) above, decide whether it is true or false. Justify your claims, providing full proofs or counterexamples, as appropriate. **[9]**

Question 4. [16 marks]

Each of the following 'proofs' has at least one fault. For each case below:

(i) state clearly what the faults are;

(ii) give a correct proof of the statement.

(a) PROPOSITION. For non-negative real numbers x, y, we have the inequality **[8]**

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

PROOF.

$$\begin{aligned}\frac{x+y}{2} \geq \sqrt{xy} &\Longrightarrow x+y \geq 2\sqrt{xy} \quad \text{(square both sides)}\\ &\Longrightarrow x^2+2xy+y^2 \geq 4xy\\ &\Longrightarrow x^2-2xy+y^2 \geq 0\\ &\Longrightarrow (x-y)^2 \geq 0.\end{aligned}$$

The last inequality is always satisfied, so we have proved the claim. □

(b) PROPOSITION. The sum of any two odd integers is even. **[8]**

PROOF. Let a and b be some odd integers, say $a = 2k-1$ and $b = 2k+1$ for some integer k. Then

$$a+b = (2k-1)+(2k+1) = 4k = 2\cdot(2k),$$

which is even, as required. □

Question 5. [16 marks]

Read the text displayed on the next two pages, and then write a report on it, comprising

- a short title [$\notin$]; **[2]**
- two/three concise key points [$\notin$]; **[4]**
- a summary of the document [$\notin$, 150]. **[10]**

End of Paper – An appendix of 2 pages follows.

This page and the next contain material[1] for Question 5.

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. The rule is known today as l'Hôpital's Rule, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2. It took considerable analysis in Section 2.4 to find $\lim_{x\to 0} \sin(x)/x$. But we have had success with the limit

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which always produces the equivalent of $0/0$ when we substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 6. **L'Hôpital's Rule** (First Form).
Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

PROOF. Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\begin{aligned}\frac{f'(a)}{g'(a)} &= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x\to a} \frac{f(x)}{g(x)}.\end{aligned}$$

EXAMPLE 1. Using L'Hôpital's Rule.

(a)
$$\lim_{x\to 0} \frac{3x - \sin(x)}{x} = \left.\frac{3 - \cos(x)}{1}\right|_{x=0} = 2.$$

(b)
$$\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \left.\frac{\frac{1}{2\sqrt{1+x}}}{1}\right|_{x=0} = \frac{1}{2}.$$

[1] Source: Thomas' Calculus. Pearson Education 2005.

Sometimes after differentiation, the new numerator and denominator both equal zero at $x = a$, as we see in Example 2. In these cases, we apply a stronger form of l'Hôpital's Rule.

THEOREM 7. **L'Hôpital's Rule** (Stronger Form).

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side exists.

Let us consider an example.

EXAMPLE 2. Applying the Stronger Form of l'Hôpital's Rule.

(a)

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} &= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \quad \text{(still 0/0; differentiate again)} \\ &= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}. \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \to 0} \frac{x - \sin(x)}{x^3} &= \lim_{x \to 0} \frac{1 - \cos(x)}{3x^2} \quad \text{(still 0/0)} \\ &= \lim_{x \to 0} \frac{\sin(x)}{6x} \quad \text{(still 0/0)} \\ &= \lim_{x \to 0} \frac{\cos(x)}{6} = \frac{1}{6}. \end{aligned}$$

End of Appendix.