Queen Mary
University of London

Main Examination period 2017

MTH5117: Mathematical Writing

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: M. Walters and L. Lacasa

Marks are deducted for incorrect grammar/spelling. In a question, or part of a question, the notation **[$\not\in$]** indicates that the answer should not contain any mathematical symbols whatsoever, apart from numerals and the three numbers e, i and π. In the absence of this notation, mathematical symbols may be used freely. As in lectures we follow the convention that $0 \in \mathbb{N}$.

Question 1. [25 marks] For each of the following mathematical objects provide two levels of description: (i) a coarse description, which only identifies the class to which an object belongs (set, function, etc.) **[$\not\in$]**; and (ii) a finer description, which describes the object in question as accurately as possible **[$\not\in$]**.

(a) $\{n \in \mathbb{N} : 2 \mid (n+1)\}$. **[5]**

(b) $(x+y)^2 = x^2 + 2xy + y^2$. **[5]**

(c) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} - 8y = \sin x$. **[5]**

(d) $\displaystyle\sum_{n=0}^{9} n^2$. **[5]**

(e) $\{(a_i)_{i=1}^{\infty} : a_i \in \mathbb{R},\ j < k \implies a_j > a_k\}$. **[5]**

Question 2. [15 marks] Each of the following 'proofs' is incorrect. In each case say why the proof is wrong.

(a) **Statement.** Let $n \in \mathbb{N}$. If n^3 is odd then n is odd.

Proof. If n is odd then $n = 2k+1$ for some $k \in \mathbb{N}$. Thus

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$$

which is odd. **[5]**

(b) **Statement.** All even natural numbers are the sum of two primes.

Proof. The number 10 is even, and it is the sum of 3 and 7, both of which are prime. **[5]**

(c) **Statement.** All two by two real matrices are invertible.

Proof. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $A' = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AA' = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Similarly $A'A = I$, so A is invertible with inverse A'. **[5]**

Question 3. [20 marks] Express each of the following statements with symbols using at least one quantifier.

(a) The natural number n is divisible by 7. **[5]**

(b) The function $f\colon \mathbb{R} \to \mathbb{R}$ is periodic with period 1. **[5]**

(c) The function $f\colon A \to B$ is injective. **[5]**

(d) The set $X \subseteq \mathbb{R}$ has a maximal element. **[5]**

Question 4. [15 marks] Each of the following statements is equivalent to a statement with an explicit implication – i.e., a statement of the form

"*For all ... x, if x ..., then ...*"

For each of the statements, write down (i) an equivalent statement with an explicit implication, (ii) its contrapositive, (iii) its converse and (iv) its negation, and (v) state whether each of these new statements is true or false (justification is not required).

(a) Every real number is a complex number. **[5]**

(b) A quadratic equation has at least one real root. **[5]**

(c) Every invertible matrix is symmetric. **[5]**

Question 5. [10 marks]

(a) Suppose that $P(n)$ is a statement for each $n \in \mathbb{N}$ and that we want to prove that $P(n)$ is true for all $n \in \mathbb{N}$. Outline the main steps in the proof method 'proof by minimal counterexample'. (In the web book this is called the 'least counterexample principle.') **[5]**

(b) Use this method to prove that every natural number greater than 1 can be written as a product of primes. **[5]**

Question 6. [15 marks] Read the text displayed on the next two pages, and then write a report on it, comprising

- a short title [$\not\in$]; **[2]**
- two concise key points [$\not\in$]; **[2]**
- a summary (approximately 150 words) of the document [$\not\in$]. **[11]**

End of Paper—An appendix of 2 pages follows.

This page and the next contain the material for Question 6

One of the basic problems of mathematics is solving equations. Using the quadratic root formula, we know how to find a point (solution) where $x^2 - 3x + 2 = 0$. There are more complicated formulas to solve cubic or quartic equations (polynomials of degree 3 or 4), but the Norwegian mathematician Niels Abel showed that no simple formulas exist to solve polynomials of degree equal to five. There is also no simple formula for solving equations like $\sin x = x^2$, which involve transcendental functions as well as polynomials or other algebraic functions.

In this extract we study a numerical method, called Newton's method or the Newton–Raphson method, which is a technique to approximate the solution to an equation $f(x) = 0$. Essentially it uses tangent lines in place of the graph of $y = f(x)$ near the points where f is zero. (A value of x where f is zero is a root of the function and a solution of the equation $f(x) = 0$.)

Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favourable circumstances, the method does the rest by moving step by step toward a point where the graph of f crosses the x-axis. At each step the method approximates a zero of f with a zero of one of its linearisations. Here is how it works.

The initial estimate, x_0, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$, to approximate the curve, and defines x_1 to be the point where the tangent meets the x-axis. The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence. We continue on, using each approximation to generate the next, until we are close enough to the root to stop. We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n, the point-slope equation for the tangent to the curve at $f(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

We can find where it crosses the x-axis by setting $y = 0$

$$\begin{aligned} 0 &= f(x_n) + f'(x_n)(x - x_n) \\ -\frac{f(x_n)}{f'(x_n)} &= x - x_n \\ x &= x_n - \frac{f(x_n)}{f'(x_n)}. \end{aligned}$$

Here is a summary of Newton's method:

(a) Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.

(b) Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{if } f'(x_n) \neq 0.$$

Convergence of Newton's Method

In practice, Newton's method usually converges with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0. You can test that you are getting closer to a zero of the function by evaluating $f(x_n)$ and check that the method is converging by evaluating $x_n - x_{n+1}$.

Theory does provide some help. A theorem from advanced calculus says that if

$$\left| \frac{f(x)f''(x)}{(f'(x))^2} \right| < 1$$

for all x in an interval about a root r, then the method will converge to r for any starting value x_0 in that interval. Note that this condition is satisfied if the graph of f is not too horizontal near where it crosses the x-axis.

Newton's method always converges if, between r and x_0, the graph of f is concave up when $f(x_0) > 0$ and concave down when $f(x_0) < 0$. In most cases, the speed of the convergence to the root r is expressed by the advanced calculus formula

$$|x_{n+1} - r| \leqslant \frac{\max|f''|}{2\min|f'|}|x_n - r|^2 = \text{constant} \cdot |x_n - r|^2,$$

where max and min refer to the maximum and minimum values in an interval surrounding r. The formula says that the error in step $n+1$ is no greater than a constant times the square of the error in step n. This may not seem like much, but think of what it says. If the constant is less than or equal to 1 and $x_n - r < 10^{-3}$, then $x_{n+1} - r < 10^{-6}$. In a single step, the method moves from three decimal places of accuracy to six, and the number of decimals of accuracy continues to double with each successive step.

The book goes on to say that this is not always the case – however, that is not part of the text you are summarising.

End of Appendix.