Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH5109: Geometry II: Knots and Surfaces

Duration: 2 hours

Date and time: 19 May 2015, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately. It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiner(s): M. Farber, B. Noohi

Question 1.

(a) State the Reidemeister theorem and describe the three types of moves which appear in the statement. [**2**]

(b) State the three main properties (axioms) of the Kauffman bracket $B(x)$ of a link diagram. [**2**]

(c) State how $B(x)$ behaves under Reidemeister moves I, II, III. [**4**]

(d) Prove the result of (c) regarding the move I. [**4**]

(e) Use the results of parts (b) and (c) to compute the Kauffman bracket $B(x)$ of the link digram shown in Figure 1: [**3**]

Figure 1: A link diagram.

Question 2.

(a) State the formula for the arc length of a parametrised plane curve $\gamma(t) = (x(t), y(t))$ where $t \in [\alpha, \beta]$. [**2**]

(b) Compute the arc length of the catenary curve

$$\gamma(t) = \left(t, a \cdot \cosh\left(\frac{t}{a}\right)\right)$$

for $t \in [0, b]$; here $a > 0$ and $b > 0$ are two positive constants. [**5**]

(c) State the formula for the curvature of a regular plane curve. [**2**]

(d) Compute the curvature of the catenary curve. [**6**]

Question 3.

(a) State the formula for curvature $k(t)$ and for torsion $\tau(t)$ of a regular space curve given in arbitrary parametrization. **[2]**

(b) Compute the curvature and torsion of the curve $\gamma(t) = (3t - t^3, 3t^2, 3t + t^3)$. **[11]**

(c) Is the curve in (b) planar? In other words, does it lie entirely in a plane $P \subset \mathbb{R}^3$? **[2]**

Question 4. Consider the surface patch

$$\boldsymbol{\sigma} = (u\cos v, u\sin v, v), \ \ -1 < u < 1, \ 0 < v < 2\pi.$$

(a) Compute the First Fundamental Form $\mathcal{F}_I$ of this surface patch. **[4]**

(b) Compute the Second Fundamental Form $\mathcal{F}_{II}$ and the mean curvature of this surface patch. **[8]**

(c) Sketch the surface patch. (Hint: what do we get if we fix v?) **[3]**

Question 5. Consider the curve $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ on a surface patch $\boldsymbol{\sigma}$.

(a) Write down (without proof) the geodesic equations for $\boldsymbol{\gamma}(t)$. **[4]**

(b) Let $\boldsymbol{\gamma}(t) = \boldsymbol{\sigma}(u(t), v(t))$ be a curve on the surface patch $\boldsymbol{\sigma} = (u\cos v, u\sin v, v)$ of Question 4. Show that if $\boldsymbol{\gamma}$ is unit-speed, then $\dot{u}^2 + (1 + u^2)\dot{v}^2 = 1$. Here dot denotes $\frac{d}{dt}$. **[3]**

(c) Show that if $\boldsymbol{\gamma}$ is a geodesic on $\boldsymbol{\sigma}$, then $\dot{v} = \frac{a}{1+u^2}$, where a is a constant. **[5]**

(d) What are the geodesics corresponding to $a = 0$? Describe exactly what they look like. **[4]**

Question 6. Let $\boldsymbol{\gamma}$ be a curve on a surface patch $\boldsymbol{\sigma}$.

(a) Write down the definitions of the geodesic curvature κ_g and the normal curvature κ_n of $\boldsymbol{\gamma}$. State Euler's formula for κ_n. [**5**]

(b) Let $\boldsymbol{\gamma}$ be a unit-speed curve on the unit sphere. Show that the normal curvature of $\boldsymbol{\gamma}$ is equal to 1 at every point. [**4**]

(c) Let $\boldsymbol{\sigma}(u, v)$ be a surface patch such that for every unit-speed curve $\boldsymbol{\gamma}$ on $\boldsymbol{\sigma}$ the normal curvature κ_n of $\boldsymbol{\gamma}$ is equal to 1 at every point. Prove that the Gauss curvature K_G is equal to 1 at every point on $\boldsymbol{\sigma}$. (Hint: you may use Euler's formula.) [**4**]

Question 7.

(a) State (without proof) the Gauss-Bonnet Theorem for a curvilinear polygon on a surface. [**5**]

(b) Let S be a surface whose Gauss curvature is everywhere equal to -1. Consider a curvilinear triangle on this surface whose edges have lengths a, b and c, and whose interior angles are α, β and γ. Suppose that the geodesic curvature of each edge is equal to 1 at every point. Show that the area of the interior of this triangle is equal to [**6**]

$$a + b + c - \alpha - \beta - \gamma + \pi.$$

End of Paper.