# MTH5105: Differential and Integral Analysis

## Duration: 2 hours

## Date and time: 11th May 2016, 14:30–16:30

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**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt ALL questions. Marks awarded are shown next to the questions.**

**Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.**

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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**Exam papers must not be removed from the examination room.**

**Examiner(s): M. Walters**

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Unless otherwise stated you may assume any standard properties of the functions sin, cos, and exp, including that they are differentiable. You should justify your answers unless otherwise stated.

**Question 1 (25 marks).**

(a) State Taylor's theorem including the Lagrange form of the remainder. **[5]**

For the rest of the question let $f\colon [-1,1] \to \mathbb{R}$ be an infinitely differentiable function satisfying

- for all $n \geqslant 0$, $f^{(n)}(0) = 1/(n+1)$ and
- for all $n \geqslant 0$ and for all $x \in [-1,1]$, $|f^{(n)}(x)| \leqslant 3$.

(b) Write down the Taylor polynomials $T_{2,0}$, $T_{3,0}$ and $T_{n,0}$. **[5]**

(c) Write down the Lagrange form of the remainder term $R_{n,0}$ and show that

$$|R_{n,0}(x)| \leqslant \frac{3|x|^{n+1}}{(n+1)!}$$

for all $n \geqslant 0$ and for all $x \in [-1,1]$. **[7]**

(d) Deduce that $T_{n,0} \to f$ pointwise on $[-1,1]$ as $n \to \infty$. **[4]**

(e) Is the convergence uniform? Briefly explain your answer. **[4]**

**Question 2 (25 marks).**

(a) State the Fundamental Theorem of Calculus. **[5]**

For the rest of the question let $f\colon [a,b] \to \mathbb{R}$ be a continuous function with $f(x) > 0$ for all $x \in \mathbb{R}$.

(b) Prove that the function $F\colon [a,b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)\mathrm{dt}$$

is continuous on $[a,b]$. Why do we know $F$ is differentiable? **[7]**

(c) Using the chain rule, or otherwise, show that the function $G\colon [a,b] \to \mathbb{R}$ defined by

$$G(x) = \exp\left(\int_a^x f(t)\mathrm{dt}\right)$$

is differentiable and find its derivative. **[7]**

(d) Show that $G^{-1}$ exists. Is $G^{-1}$ differentiable? Briefly justify your answer. **[6]**

**Question 3 (25 marks).**

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function and $P$ be a partition of $[a,b]$.

(a) Define the **upper and lower sums** $U(f,P)$ and $L(f,P)$. **[4]**

(b) Let $g$ be the function $g : [0,4] \to \mathbb{R}$ given by the graph below, and let $P$ be the partition $\{0,1,2,4\}$. Find $U(g,P)$ and $L(g,P)$ in this case. **[4]**

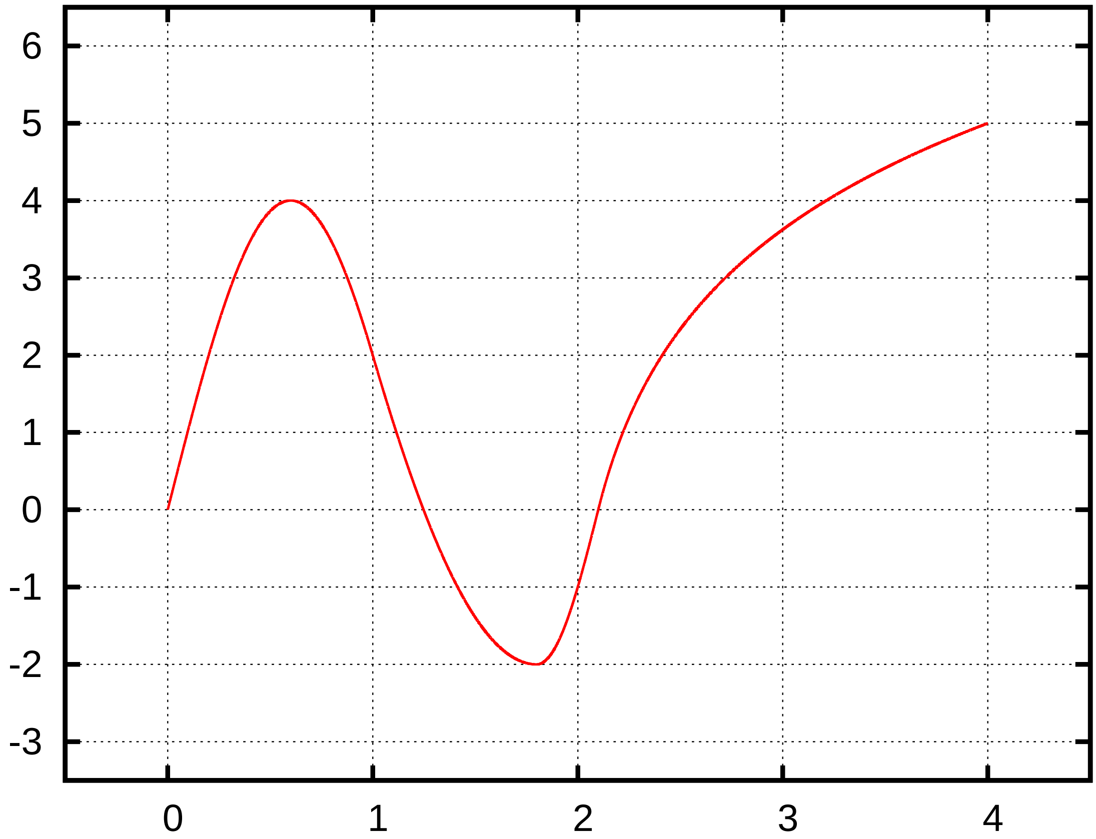


Figure 1: The function $g$.

(c) Starting from the lower and upper sums you defined in part (a), give the definition that $f$ is **integrable** and define $\int_a^b f$ when it exists. **[4]**

(d) State the Riemann integrability condition. **[4]**

(e) Suppose that $f$ is increasing. Using the Riemann integrability condition, prove that $\int_a^b f$ exists in this case. **[5]**

(f) Give an example of a bounded function $f\colon [a,b] \to \mathbb{R}$ that is not integrable. Briefly justify that it is not integrable. **[4]**

**Question 4 (25 marks).**

(a) State the definition that a function $f\colon \mathbb{R} \to \mathbb{R}$ is **differentiable** at a point $a$. [**5**]

(b) Give an example of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that is not differentiable at zero (i.e., $f'(0)$ does not exist) and justify your answer. (Your function must be continuous but you do **not** need to justify the continuity.) [**5**]

(c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying $|f(x)| \leqslant x^2$ for all $x$. Prove that $f$ is differentiable at zero (i.e., that $f'(0)$ exists). [**5**]

(d) State the Mean Value Theorem. [**5**]

(e) Let $f$ and $g$ be differentiable functions $f, g, : [a, b] \to \mathbb{R}$ such that $f(a) = g(a)$, and $f(b) = g(b)$. Prove that there exists $c \in (a, b)$ with $f'(c) = g'(c)$. [**5**]

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**End of Paper.**