Queen Mary
University of London

Main Examination period 2017

MTH5104: Convergence and Continuity

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: R. Buzano, G. Bianconi

You may assume any standard properties of the sine, cosine and exponential functions including the fact that they are continuous.

Question 1. [20 marks]
Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $x \in \mathbb{R}$.

(a) Define (using quantifier expressions) what it means for $(x_n)_{n=1}^{\infty}$ to converge to x. **[3]**

(b) Define (using quantifier expressions) what it means for x to be an accumulation point of $(x_n)_{n=1}^{\infty}$. **[3]**

Now let $(x_n)_{n=1}^{\infty}$ be the sequence defined by $x_n = (-1)^n\left(1 - \frac{1}{n}\right)$.

(c) Prove directly from the definition that $(x_n)_{n=1}^{\infty}$ does **not** converge to any real number. **[5]**

(d) Prove directly from the definition that $x = 1$ is an accumulation point of $(x_n)_{n=1}^{\infty}$. **[4]**

(e) Prove that $x = 0.9$ is **not** an accumulation point of $(x_n)_{n=1}^{\infty}$. **[5]**

Question 2. [20 marks]
Let $\alpha > 3$ be a real number. Define the sequence $(x_n)_{n=1}^{\infty}$ inductively by $x_1 = \alpha$ and

$$x_{n+1} = \sqrt{\alpha + 2x_n}, \qquad \forall n \in \mathbb{N}.$$

(a) Prove that the sequence $(x_n)_{n=1}^{\infty}$ is strictly decreasing. **[6]**

(b) Prove that the sequence $(x_n)_{n=1}^{\infty}$ is bounded below by $\sqrt{\alpha}$. **[6]**

(c) Prove that the sequence $(x_n)_{n=1}^{\infty}$ converges. **[3]**

(d) Find, with justification, the limit of $(x_n)_{n=1}^{\infty}$. **[5]**

Question 3. [20 marks]

(a) Which of the following series converge? Justify your answers. (You may use any results from the course provided you state clearly which result you are using.)

$$(i)\ \sum_{k=1}^{\infty} \frac{k^3}{k^5+3}, \qquad (ii)\ \sum_{k=1}^{\infty} \frac{3^k}{5^k+3}, \qquad (iii)\ \sum_{k=1}^{\infty} \frac{\cos(\frac{1}{k})}{(-1)^k}.$$ **[12]**

(b) Find the value of the series $\sum_{k=1}^{\infty} x_k$ given by $x_k = \dfrac{1+2^k}{3^k}$. **[5]**

(c) Let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection. Is it true or not that $\sum_{k=1}^{\infty} x_{\phi(k)} = \sum_{k=1}^{\infty} x_k$ for the series given in part (b)? Briefly justify your answer. **[3]**

Question 4. [20 marks]

(a) Define (using quantifier expressions) what it means to say that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** at a point $a \in \mathbb{R}$. **[3]**

(b) Prove directly from the definition that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 - x$ is continuous at all $a \in \mathbb{R}$. **[7]**

(c) Give the negation of your quantifier statement from part (a), i.e. define what it means for f to **not** be continuous at $a \in \mathbb{R}$. **[3]**

(d) For $\beta \in \mathbb{R}$, define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \beta & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Prove that for all $\beta \in \mathbb{R}$, f is **not** continuous at $a = 0$. **[7]**

Question 5. [20 marks]
Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5\sin^4(x) - \cos(x)$.

(a) State the Intermediate Value Theorem. **[3]**

(b) Prove that the equation $f(c) = 0$ has a solution c in $[0, \pi]$. **[5]**

(c) Prove that f has a fixed point in $[0, \pi]$. **[5]**

(d) Prove the following statement: for every $\varepsilon > 0$, there exists an open interval $(a, b) \subset [0, \pi]$, such that for all $c \in (a, b)$ we have $|f(c)^3 + 3f(c) + 3| < \varepsilon$. **[7]**

End of Paper.