Queen Mary
University of London

MTH5102: Calculus III

Duration: 2 hours

Date and time: 16 May 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): L. Lacasa

Question 1 (10 marks). Let $\mathcal{C}$ be the curve in $\mathbf{R}^3$ whose parametric equation reads $\mathbf{r}(t) = (\frac{t}{2}, \frac{t^2}{2}, \frac{2t^{3/2}}{3})$. Consider the points $A = (0,0,0)$, $B = (1/2, 1/2, 2/3)$ and $C = (1, 2, 8/3)$.

(a) Show that $A,\ B$ are in the curve and determine whether or not C is on the curve (justify your answer). **[2]**

(b) Compute the arc of length of $\mathcal{C}$ between the points A and B. **[8]**

Question 2 (25 marks). Let $\mathcal{C}$ be an ellipse in the XY plane described in implicit form by $x^2 + 4y^2 = 1$, and let $\mathbf{F} = x\mathbf{i} + \mathbf{j}$ be a vector field.

(a) Find a parametric equation for $\mathcal{C}$. **[3]**

(b) Compute the line integral of $\mathbf{F}$ along $\mathcal{C}$ (travelled anticlockwise) from $A = (1,0)$ to $B = (-1,0)$. **[7]**

(c) Make a sketch of $\mathbf{F}$ and use the sketch to deduce the value of $\nabla \times \mathbf{F}$. **[5]**

(d) State Stokes's theorem and use this along with previous part of the question to deduce the line integral of $\mathbf{F}$ along $\mathcal{C}$ (travelled anticlockwise) from B to A. Give as many details as needed. **[10]**

Question 3 (20 marks). Let $\mathcal{S}$ be a sphere of radius a centred at the origin and let $\mathbf{F} = z\mathbf{k}$ be a vector field.

(a) Give a parametric equation for $\mathcal{S}$ along with the range of the parameters. **[5]**

(b) State the divergence theorem and use it, explaining why it can be applied, to compute the flux of $\mathbf{F}$ across $\mathcal{S}$. **[5]**

(c) Express $\mathbf{F}$ and $d\mathbf{S}$ in spherical coordinates. **[6]**

(d) Using the spherical coordinate system, compute the flux of $\mathbf{F}$ across $\mathcal{S}$. **[4]**

Question 4 (15 marks). Let $f(x)$ be a periodic function of period 2π defined in $(-\pi, \pi]$ by $f(x) = 2x$.

(a) State the formula for the Fourier series $S(x)$ of $f(x)$. **[2]**

(b) State (without proof) Dirichlet's theorem, and use it to compute $S(\pi)$. What does this reflect about the behaviour of $f(x)$ at $x = \pi$? **[4]**

(c) Calculate the Fourier series $S(x)$ of $f(x)$. **[9]**

Question 5 (15 marks). Consider the vector field $\mathbf{F} = (y, x - 2y, 1)$.

(a) Prove that $\mathbf{F}$ is conservative and compute a scalar potential ϕ for $\mathbf{F}$. **[8]**

(b) Prove that if $\mathbf{F}$ is also solenoidal then its scalar potential ϕ fulfils Laplace's equation, and check that the scalar potential ϕ computed above does not fulfil Laplace's equation. **[7]**

Question 6 (15 marks). (a) State Laplace's equation in two dimensional Cartesian coordinates. **[2]**

(b) Justify the following: if $f(x, y)$ and $g(x, y)$ are two harmonic functions then their sum is also a harmonic function, but their product is not necessarily a harmonic function (by giving counterexample). **[4]**

(c) Explain the method of separation of variables, providing as many details as needed. **[9]**

End of Paper.