Queen Mary University of London

B. Sc. Examination by course unit 2014

MTH5102 Calculus III

Duration: 2 hours

Date and time: 19 May 2014, 10:00 am

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You should attempt all questions. Marks awarded are shown next to the questions.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): L. Lacasa

Question 1 [22]

Let $\mathcal{C}$ a curve in $\mathbf{R}^3$ whose parametric equation reads $\mathbf{r}(t) = (\cos t, \sin t, 2t)$, and consider the points A, B, C and D whose coordinates are, respectively, $A = (1, 0, 0)$, $B = (0, 1, \pi)$, $C = (1, 1, 1)$ and $D = (-1, 0, 2\pi)$.

(a) Describe what kind of curve $\mathcal{C}$ is, and make a sketch. [4]

(b) Briefly justify which points from $\{A, B, C, D\}$ belong to the curve $\mathcal{C}$. [4]

(c) Calculate the arc length of $\mathcal{C}$, from A to D. [6]

(d) Consider the vector field $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$. Calculate the line integral of $\mathbf{F}$ over $\mathcal{C}$, between A and D. [8]

Question 2 [21]

Let U be a suitably differentiable scalar field, and $\mathbf{F}$ be a suitably differentiable vector field.

(a) Write down the expressions for ∇U, $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ in Cartesian coordinates. [3]

(b) For each of the following expressions, state (without proof) whether it is (i) a scalar field (ii) a vector field or (iii) not a valid expression:

$$U(\nabla \times \mathbf{F})\ ; \quad U(\nabla \cdot \mathbf{F})\ ; \quad \nabla \times (\nabla U)\ ; \quad \nabla \times (\nabla \cdot \mathbf{F})\ .$$

[4]

(c) Let $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ a vector field. Prove that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$. [5]

Let $\mathbf{F} = (y + z)\mathbf{i} + x\mathbf{j} + (x + 2z)\mathbf{k}$.

(d) Is $\mathbf{F}$ solenoidal? Justify your answer. [2]

(e) Give the definition of a scalar potential, saying what condition is needed for a scalar potential to exist. Show that this condition holds for $\mathbf{F}$, and calculate a scalar potential for $\mathbf{F}$. [7]

Question 3 [12]

(a) State (without proof) the Divergence theorem (define the terms used and any required conditions). [4]

(b) Apply this theorem to calculate the surface integral of the vector field $\mathbf{F} = y\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$ over a cylinder such that $x^2 + y^2 < a^2$, $0 < z < b$, where a, b are positive constants. Explain all the assumptions you make to apply the theorem. [8]

Question 4 [11]

Consider the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

(a) Express the cartesian coordinates (x, y, z) in terms of spherical coordinates (r, θ, ϕ). [2]

(b) Express the spherical coordinate unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ in terms of the cartesian unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. [3]

(c) Consider the vector field $F = -z\mathbf{k}$ in the surface of a sphere of radius a with vector area element $d\mathbf{S}$. Calculate the scalar product $\mathbf{F} \cdot d\mathbf{S}$. [6]

Question 5 [16]

Let $f(x)$ be a periodic function of period 2π defined in $(-\pi, \pi)$ by

$$f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi. \end{cases}$$

(a) State (without proof) the general expression for a Fourier series $S(x)$. [2]

(b) Show (without calculating the terms) that the Fourier series of $f(x)$ does not have any $\cos nx$ terms. [3]

(c) Calculate its Fourier series. [7]

(d) By suitably evaluating the Fourier series and by virtue of Dirichlet's theorem, calculate the sum $\sum_{k=0}^{\infty} (-1)^k/(2k+1)$. [4]

Question 6 [9]

Consider the Laplace equation $\nabla^2 \phi = 0$ over a rectangle in the XY plane, subject to some boundary conditions.

Briefly explain the method of separation of variables, and justify why this method proposes that a possible solution to the problem takes the form

$$\phi(x, y) = (A\cos(kx) + B\sin(kx))(C\cosh(ky) + D\sinh(ky)),$$

where A, B, C, D are constants that depend on the boundary conditions.

Question 7 [9]

The Milky Way (see picture below) is a rotating galaxy. Let us define a function **V** that describes the instantaneous velocity of a planet. The location of this planet is referenced to the centre of the galaxy, where we put the origin of the coordinate system.

(a) Justify why **V** is a vector field. [2]

(b) Explain how could you describe the average direction and magnitude of rotation of the Milky Way close to its centre. [4]

(c) Assuming that the Milky Way is approximately a planar galaxy which belongs to the XY plane, calculate the average direction of the rotation close to its centre. [3]

End of Paper