Queen Mary
University of London

MTH5100: Algebraic Structures I

Duration: 2 hours

Date and time: 23 May 2016, 14.30h–16.30h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): L. H. Soicher

Question 1.

(a) Define what it means for a ring R to be a *ring with identity.* [**3**]

(b) Define what it means for a ring R to be a *commutative ring.* [**3**]

(c) Define what is meant by a *unit* in a ring with identity. [**3**]

(d) Define what is meant by a *zero-divisor* in a commutative ring. [**3**]

(e) Define what is meant by an *integral domain.* [**3**]

(f) Define what is meant by a *field.* [**3**]

(g) Define what is meant by a *subring* S of a ring R. [**3**]

(h) Define what is meant by an *ideal* I of a ring R. [**3**]

Question 2. Let T be the subring $\{[0]_6, [2]_6, [4]_6\}$ of $\mathbb{Z}_6$. [You are not required to prove that T is a subring of $\mathbb{Z}_6$.]

(a) Is T a ring with identity? Justify your answer. [**4**]

(b) Is T a field? Justify your answer. [**4**]

(c) Is $\mathbb{Z}_6$ a field? Justify your answer. [**4**]

Question 3.

(a) Write down the units in $\mathbb{Z}_{15}$. [**3**]

(b) Write down the zero-divisors in $\mathbb{Z}_{15}$. [**3**]

Question 4. Let X be a set and let $\mathcal{P}(X)$ denote the Boolean ring whose elements are the subsets of X, with addition being symmetric difference and multiplication being intersection. [You do not have to prove that $\mathcal{P}(X)$ is a ring.]

(a) What is the zero-element of $\mathcal{P}(X)$? [**2**]

(b) Is $\mathcal{P}(\{1\})$ an integral domain? Justify your answer. [**4**]

(c) Is $\mathcal{P}(\{1, 2\})$ an integral domain? Justify your answer. [**4**]

Question 5. Let R and S be rings, and let $\theta : R \to S$ be a homomorphism.

(a) Define what is meant by the *image* $\mathrm{Im}(\theta)$ and by the *kernel* $\mathrm{Ker}(\theta)$, of θ. [**4**]

(b) Apply one of the subring tests to prove that $\mathrm{Im}(\theta)$ is a subring of S. [You may make use of any basic properties of homomorphisms proved in lectures.] [**6**]

(c) Apply the ideal test to prove that $\mathrm{Ker}(\theta)$ is an ideal of R. [You may make use of any basic properties of homomorphisms proved in lectures.] [**6**]

Question 6.

(a) Define what it means to be an *irreducible* element of an integral domain R. [**3**]

(b) Let $S = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. Prove that $1 + \sqrt{-5}$ is an irreducible element of S. [You may assume, without proof, that S is an integral domain, and that the only units in S are 1 and -1.] [**7**]

Question 7.

(a) Let R be a commutative ring with identity. Prove that if $\{0\}$ and R are the only ideals of R then R is a field. [You may assume, without proof, that if $a \in R$, then $\langle a \rangle = aR$ is an ideal of R.] [**4**]

(b) Let R be a commutative ring with identity, and let I be an ideal of R. Apply the Second Isomorphism Theorem to prove that if I is a maximal ideal of R then R/I is a field. [**6**]

Question 8. In this question we consider the ring $\mathbb{R}[x]$ of polynomials with real number coefficients. [You may assume, without proof, that $\mathbb{R}[x]$ is an integral domain, and may apply any results from this module in your justifications.]

(a) Is $\mathbb{R}[x]$ a principal ideal domain? Justify your answer. [**4**]

(b) Is the factor ring $\mathbb{R}[x]/\langle x^2 - 1\rangle$ an integral domain? Justify your answer. [**4**]

(c) Is the factor ring $\mathbb{R}[x]/\langle x^2 + 1\rangle$ an integral domain? Justify your answer. [**4**]

End of Paper.