Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH5100 Algebraic Structures I

Duration: 2 hours

Date and time: 7 May 2014, 14.30h–16.30h

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You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorized use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Examiner(s): L. H. Soicher

Question 1 State in full the definition of a *ring*. [8]

Question 2 (a) Let $X = \{1, 2, 3, 4\}$ and consider the Boolean ring $P = \mathcal{P}(X)$ consisting of the subsets of X, where addition is symmetric difference and multiplication is intersection.

(i) What is the zero-element in P? [2]

(ii) Let $x = \{1, 4\}$. Write down all the elements $y \in P$ with the property that $xy = 0$ in P. [4]

(b) Let B be any ring with the property that $x^2 = x$ for all $x \in B$ (where x^2 means xx). Prove that $x = -x$ for all $x \in B$. [4]

Question 3 (a) For each of the following statements, write down whether it is true or false. You do not need to justify your answer. [5]

(i) The ring $6\mathbb{Z}$ has cardinality 6.

(ii) The set $0\mathbb{Z}$ is a subring of $\mathbb{Z}$.

(iii) The empty set is a subring of every ring.

(iv) Every ring has at least two subrings.

(v) The ring $2\mathbb{Z}$ has exactly two subrings.

(b) Write down representatives for the distinct cosets of $20\mathbb{Z}$ in $4\mathbb{Z}$. [You may assume, without proof, that $20\mathbb{Z}$ is a subring of $4\mathbb{Z}$.] [5]

Question 4 Let R be a ring with identity. Recall that an element $u \in R$ is called a *unit* if there is an element $v \in R$ with $uv = vu = 1$.

(a) Let R be a ring with identity and let u and v be units in R. Prove that all the elements 1, u^{-1} and uv are units in R. [4]

(b) Let R be a ring with identity, and define a relation A on R by $(a, b) \in A$ if and only if $b = au$ for some unit $u \in R$. Prove that A is an equivalence relation on R. [6]

Question 5 Let R and S be rings, and let $\theta : R \to S$ be a homomorphism.

(a) Define what is meant by the *image* $\mathrm{Im}(\theta)$ and by the *kernel* $\mathrm{Ker}(\theta)$, of θ. [4]

(b) Apply the Ideal Test to prove that $\mathrm{Ker}(\theta)$ is an ideal of R. [You may assume, without proof, the basic properties of homomorphisms proved in lectures.] [5]

(c) Let $a, b \in R$. Prove that a and b are in the same coset of $\mathrm{Ker}(\theta)$ in R if and only if $a\theta = b\theta$. [8]

Question 6 (a) Define what is meant by a *zero-divisor* in a ring. [3]

(b) Write down all the zero-divisors in $\mathbb{Z}/18\mathbb{Z}$ (also known as $\mathbb{Z}_{18}$). [4]

(c) Prove that if u is a unit in a ring R with identity then u is not a zero-divisor in R. [4]

Question 7 Let $S = \{a + bi : a, b \in \mathbb{Z}\}$ (where $i \in \mathbb{C}$, $i^2 = -1$).

(a) Apply a Subring Test to prove that S is a subring of the ring $\mathbb{C}$ of complex numbers. [5]

(b) Explain why S is an integral domain. [3]

(c) Determine a factorisation of the integer 5 into irreducible elements of S. Justify your answer. [You may assume, without proof, that the units in S are $1, -1, i, -i$.] [8]

Question 8 (a) Define what is meant by a *field*. [3]

(b) Prove that if F is a field then the only ideals of F are $\{0\}$ and F. [6]

(c) Define what is meant by a *maximal ideal* of a ring R. [3]

(d) Let R be a ring and let I be an ideal of R. Apply the Second Isomorphism Theorem to prove that if R/I is a field then I must be a maximal ideal of R. [6]

End of Paper