Queen Mary
University of London

MTH4110: Mathematical Structures

Duration: 2 hours

Date and time: 3rd May 2016, 10:00–12:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): O. F. Bandtlow

Question 1. Let $n \in \mathbb{N}$. Using induction, prove that

$$(1-x)\sum_{k=0}^{n-1} x^k = 1 - x^n$$

for any real number x. **[12]**

Question 2.

(a) Give the definition of a *prime number*. **[4]**

(b) Given any natural number n is there a prime number larger than n^n? Give reasons for your answer. You are allowed to quote results proved in the lectures. **[4]**

(c) Let n be a natural number. Prove, using the equation in Question 1 or otherwise, that if $2^n - 1$ is prime, then n is prime. **[8]**

Question 3. Let A, B, and C be sets.

(a) Explain what is meant for A to be a *subset* of B. **[4]**

(b) Show that if $A \subseteq B$, $B \subseteq C$ and $C \subseteq A$, then $A = B = C$. **[4]**

(c) Is it possible that $A \in B$ and $A \subseteq B$? Give reasons for your answer. **[4]**

Question 4.

(a) Let A and B be sets, and let $f : A \to B$ be a function. Explain what is meant by saying that f is

(i) *injective*, (ii) *surjective*, (iii) *bijective*. **[6]**

(b) Is there a bijection from $\mathbb{Z}$ to $\mathbb{R}$? Give a brief explanation for your answer. **[4]**

(c) Let $\mathscr{F}(\mathbb{N})$ denote the set of all finite subsets of $\mathbb{N}$ and let the function $f : \mathscr{F}(\mathbb{N}) \to \mathbb{N}$ be given by $f(A) = |\mathscr{P}(A)|$, where $\mathscr{P}(A)$ denotes the power set of A. Is f injective? Is it surjective? Give reasons for your answers. **[6]**

Question 5.

(a) State the Binomial Theorem. [**4**]

(b) Let n be a natural number. Show that n^3 is even if and only if n is even. [**4**]

(c) Prove that $\sqrt[3]{2}$ is irrational. [**6**]

Question 6. Let $F : \mathbb{C} \to \mathbb{C}$ be a function. A complex number w is said to be a *fixed point* of F, if $F(w) = w$.

(a) Show that $2 - \mathrm{i}$ is a fixed point of the function f given by $f(z) = z^2 - 3z + 5$. [**4**]

(b) Find all fixed points of the function g given by $g(z) = z^3 + z + 8\mathrm{i}$. [**6**]

(c) Does every polynomial of degree greater 2 have a fixed point? Give reasons for your answer. [**4**]

Question 7. Let n be a natural number greater 1 and let $x_1, x_2, \ldots, x_n$ be real numbers. Consider the following statement.

If the product $x_1 x_2 \cdots x_n$ is zero, then x_1 and x_n are zero.

(a) Write down the contrapositive. [**3**]

(b) Write down the converse. [**3**]

(c) Is the statement true? Is the contrapositive true? Is the converse true? Give reasons for your answer. [**6**]

Question 8. Find the flaw in the following proof. [**4**]

Theorem 1 *is the largest natural number.*

Proof The proof is by contradiction. Let n be the largest natural number, and suppose that $n > 1$. Multiplying both sides of this inequality by n we see that $n^2 > n$. Thus n^2 is a natural number greater than n, contradicting the fact that n is to be the largest natural number. So the assumption $n > 1$ is wrong, and we have shown that $n = 1$. So 1 is the largest natural number. □

End of Paper.