Queen Mary
University of London

Main Examination period 2017

MTH4107-4207: Introduction to Probability

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: W. Mannan, F. Vivaldi

In this paper we employ the following standard notation: Ω is a sample space, ω is an outcome, $\mathbb{P}$ is a probability, A,B,C are events, A^c is the complement of A, $|A|$ is the cardinality of A.

Question 1. [18 marks] (BASICS)

(a) We select three integers from a set of integers, and we consider the events: **[6]**

A: at least one selected integer is odd
B: at most one selected integer is odd.

Describe each of the following events by a sentence:

$$A \cap B, \qquad A \setminus B, \qquad B \setminus A.$$

(b) Two of the following expressions are meaningless; identify the meaningless expressions, and explain why they are meaningless. **[6]**
[*No marks will be allocated without a correct explanation, or if there are more than two selections.*]

1. The intersection of two events.
2. The complement of the sample space.
3. The square of the probability of an event.
4. The probability of the square of an event.
5. An infinite event.
6. The first outcome of an event.

(c) Two of the following sentences are false; identify the false sentences, and explain why they are false. **[6]**
[*Same marking scheme as part* (b).]

1. The sample space is an event.
2. A probability is a function.
3. All events contain outcomes.
4. Every outcome is an elementary event.
5. Every outcome is an element of some event.
6. The symmetric difference of two events is an event.

Question 2. [14 marks] (SAMPLING)

(a) When I open a bank account, I am allocated a random 4-digit personal identification number (which may begin with one or more zeros). What is the probability that my number has no digits occurring more than once? **[6]**

(b) A box of 20 spare parts contains 15 good parts and 5 defective ones. If four parts are selected at random from this box, what is the probability that exactly k of them will be good, as a function of k $(0 \leqslant k \leqslant 4)$? **[8]**

[*There is no need to simplify answers. Some explanation is needed for full marks.*]

Question 3. [24 marks] (CONDITIONAL PROBABILITY)

(a) Let A, B, C be events with $\mathbb{P}(A), \mathbb{P}(B), \mathbb{P}(C) > 0$.

(i) Compute $\mathbb{P}(A|B)$ in the following special cases: **[6]**

1) B is a subset of A; 2) A and B partition Ω.

(ii) Assuming that A and B are independent, express $\mathbb{P}(A \cap B \cap C)$ as a product of some of the following probabilities: **[6]**

$$\mathbb{P}(A) \quad \mathbb{P}(B) \quad \mathbb{P}(A|B \cap C) \quad \mathbb{P}(A|C)$$
$$\mathbb{P}(C) \quad \mathbb{P}(C|A) \quad \mathbb{P}(C|B \cap A).$$

(iii) Prove that $\mathbb{P}(A^c|B) + \mathbb{P}(A|B) = 1$. **[6]**

(b) My flight has a 40% chance of being late, and if it's late, then there is a 90% chance that I'll miss the last train from the airport. But even if the flight is not late, there is still a 20% chance that I'll miss the last train due to a queue at passport control. If I miss the last train, what is the probability that the flight was late? **[6]**

[*For full marks you must state precisely any result you make use of.*]

Question 4. [25 marks] (RANDOM VARIABLES)

(a) Let X be a random variable. [**5**]

(i) Explain what it means to say that X is discrete.

(ii) Represent the event '$X = 1$' as a set, using only symbols.

(b) I toss a fair coin three times. Let X be the number of times a toss gives the same outcome as the previous toss.

(i) Compute the expectation of X. [**8**]

(ii) Compute the variance of the random variable X^2. [**6**]

Explain what you do.

(c) A continuous random variable has the following probability density function: [**6**]

$$\delta(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant \sqrt{2} \\ 0 & \text{otherwise.} \end{cases}$$

Compute the expectation and the median of such a random variable: which of the two is the largest?

Question 5. [19 marks] (DISTRIBUTIONS)

(a) Compute and draw the cumulative distribution function of the Binomial$(2, 1/3)$ distribution. [**6**]

(b) Let $\Omega = \{-1, 0, 1\}$ and let

$$\mathbb{P}(-1) = 1/2 \qquad \mathbb{P}(0) = 1/6 \qquad \mathbb{P}(1) = 1/3.$$

We form samples by selecting elements of Ω according to the above probabilities, each selection being independent from all other selections.

(i) Determine the distribution of the following random variables: [**4**]

1) The number X of non-zero entries in a sample of length n.

2) The number Y of entries up to and including the first strictly positive entry.

(ii) With Y as above, determine $\mathbb{P}(Y \geqslant 3)$. [**4**]

(c) Explain the appearance of the exponential function in the derivation of the Poisson distribution. [**5**]
[*A good presentation is required for full marks.*]

End of Paper.