Queen Mary
University of London

B. Sc. Examination by course unit 2014

MTH4107: Introduction to Probability

Duration: 2 hours

Date and time: 6 May 2014, 2:30 pm

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt all questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Robert Johnson

Question 1 [10]

A student is randomly selected from those attending a lecture. Let B be the event "the chosen student travelled to the lecture by Bus", U be the event "the chosen student travelled to the lecture by Underground", and C be the event "the chosen student cycled to the lecture". Suppose that no student used more than one of these means of transport and that $\mathbb{P}(B) = 3/10$, $\mathbb{P}(U) = 1/2$, $\mathbb{P}(C) = 1/10$. Let L be the event that "the chosen student was late to the lecture" and suppose that $\mathbb{P}(L) = 3/8$ and $\mathbb{P}(L \cap B) = 1/8$.

(a) Write down in words what the event $B \cup U$ is and find its probability.

(b) Write down in symbols what the event "the chosen student was late and did not come by Bus" is and find its probability.

(c) Do the events B, U, C partition the sample space? What does your answer mean in this context?

Question 2 [10]

For some fixed $n \geq 3$, I write down a sequence of n letters each of which is a, b or c at random with all possibilities equally likely.

(a) What is the cardinality of the sample space?

(b) What is the probability that my sequence does not contain the letter a?

(c) What is the probability that my sequence contains exactly one a?

(d) What is the probability that no letter occurs more than once in my sequence?

[Hint: Consider the cases $n = 3$ and $n > 3$ separately.]

Question 3 [10]

Amanda and Brian are packing boxes to be sent out from a warehouse. Suppose that Amanda packs 1/3 of the boxes and makes a mistake in 1/10 of those she packs. Brian packs the remaining 2/3 of the boxes and makes a mistake in 1/5 of those he packs.

(a) A box is chosen at random. Calculate the probability that a mistake was made in packing it.

(b) Suppose that a mistake is found, calculate the probability that the box was packed by Amanda.

(c) The manager wants to redistribute the work to ensure that there are mistakes with at most 1/8 of the boxes. What is the smallest proportion of boxes which should be packed by Amanda to ensure this?

Question 4 [10]

(a) Describe how the $\mathrm{Bin}(n, p)$ distribution arises from a sequence of Bernoulli trials.

(b) Let H be the number of Heads seen when I toss 4 fair coins. State the distribution of H and its expectation and variance.

(c) I perform the experiment of tossing 4 fair coins repeatedly until I get the outcome 0 heads. Let X be the number of times I perform the experiment. State the distribution of X and its expectation and variance.

Question 5 [10]

I make an unordered selection of r objects from a set of n objects without repetition.

(a) State and prove a formula for the number of ways of making such a selection.

Suppose now that $n = 10$ and $r = 3$. Suppose also that when I make a selection all possibilities are equally likely.

(b) Calculate the number of such selections.

(c) Having made such a selection, I replace all objects and make a second selection of 3 things without repetition. What is the probability that in my second selection I do not see any of the 3 objects in my first selection?

Question 6 [10]

Let U and V be discrete random variables with probability mass functions given by:

u	0	1
$\mathbb{P}(U = u)$	1/2	1/2

v	0	1	2
$\mathbb{P}(V = v)$	1/3	1/3	1/3

Find the joint probability mass function of U and V and their covariance under each of the following assumptions:

(a) U and V are independent random variables,

(b) $\mathbb{P}(U = 0, V = 0) = \mathbb{P}(U = 1, V = 2) = 0$.

Question 7 [20]

Let X be a continuous random variable with probability density function f_X.

(a) Define the expectation of X.

An archer fires an arrow which hits a circular target of radius 2 metres. Suppose that the arrow is equally likely to hit any point of the target in the sense that the probability that it lies in a given region is proportional to the area of the region. Let A be the distance from the centre of the target to the point the arrow hits.

(b) Find the cumulative distribution function (cdf) of A.

(c) Find the probability density function (pdf) of A.

(d) Calculate the expectation of A.

(e) Find the cdf of the random variable $2 - A$.

(f) Suppose that if the arrow is within 0.5 metres of the centre the archer scores 10 points, if the arrow is between 0.5 and 1 metre from the centre the archer scores 5 points, and otherwise they score 0 points. Describe the random variable "the number of points scored" and find its expectation.

Question 8 [20]

(a) Define what it means for the two events A and B to be independent.

(b) Define what it means for the three events A, B and C to be mutually independent.

(c) Suppose that A and B are independent, must A and B^c be independent? Justify your answer.

(d) Suppose that A, B and C are mutually independent, must $A \cap B$ and $B \cap C$ be independent? Justify your answer.

I roll a fair 6-sided die twice. Let O be the event that the first roll is odd, T be the event that the sum of the two rolls is odd, and P be the event that the product of the two rolls is odd.

(e) Are O and T independent? Are O and P independent? Justify your answers.

(f) Use the result of part (c) to deduce three consequences of part (e) stated in words.

End of Paper