Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH4104: Introduction to Algebra

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **3 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

The exam is closed-book, and **no outside notes are allowed**.

Only approved non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: A. Fink, F. Rincón

Question 1 [16 marks].

(a) Use Euclid's Algorithm to find the greatest common divisor of 165 and 37. Show all your working. **[8]**

(b) Use the Extended Euclidean Algorithm to compute the multiplicative inverse of $[37]_{165}$. Show all your working. **[8]**

Question 2 [14 marks].

(a) Let X be a set. Define what it means for R to be a **relation** on X. **[3]**

Now let R be the relation on the set $\mathbb{R}[x]$ of polynomials with real coefficients defined by

$$(f, g) \in R \text{ if and only if } x^2 + 1 \text{ divides } g - f.$$

(b) Prove that R is transitive. **[5]**

For part (c) below, you may assume that R is an equivalence relation.

(c) Find a polynomial h with $\deg h \leqslant 1$ such that h is an element of the equivalence class $[x^3 + 2x^2 + 3x + 4]_R$. **[6]**

Question 3 [13 marks].

(a) State the definition of a **partition** of a set X. **[4]**

(b) Let R be an equivalence relation on X. The Equivalence Relation Theorem describes how to use R to produce a partition P of X. Write down how P is defined in terms of R. Your answer should include how the parts of P are defined [i.e. don't just use a symbol for each part without explaining what it means]. **[4]**

(c) Give an example of a partition of $\mathbb{C}$ with exactly two parts. **[2]**

(d) Write down all possible partitions of the set $\{1, 2\}$. **[3]**

Question 4 [10 marks].

(a) Let R be a ring. In order to be a **field**, R must satisfy four further axioms. Write down the names of these axioms. You need not write out what the axioms assert. **[4]**

(b) The ring $\mathbb{Z}_{35}$ is not a field. Name one of the axioms from your list in part (a) that is not satisfied by $\mathbb{Z}_{35}$. Write down a counterexample to this axiom in $\mathbb{Z}_{35}$, and explain why your counterexample is valid. **[6]**

Question 5 [14 marks]. Let U be the set $\{ (a,b,c) : a,b,c \in \mathbb{R} \}$. Two operations $+$ and $\cdot$ on U are defined by

$$(a,b,c) + (d,e,f) = (a+d, b+e, c+f),$$
$$(a,b,c) \cdot (d,e,f) = (ad, ae+bf, cf).$$

(a) Prove the multiplicative identity law for U. **[8]**

(b) Prove that $(2,1,0)$ is not a unit in U. **[6]**

Question 6 [15 marks].

(a) Define what it means for a set G with a binary operation $*$ to be a **group**. Include the full statements of the axioms. **[5]**

Now let $G = \{ x \in \mathbb{R} : x > -1 \}$, with the binary operation $*$ given by $a * b = ab + a + b$.

(b) Prove that $*$ is associative. **[6]**

(c) Write down the identity element of $(G, *)$, and a formula for the inverse of an element $x \in G$. You need not provide the proofs. **[4]**

Question 7 [18 marks].

(a) Let n be a positive integer. Define the **symmetric group** S_n. Make sure to specify both the set S_n and the definition of its group operation. **[4]**

(b) How many elements does S_n have? **[2]**

Now let g be the element

$$(1\,4\,6\,9)(2\,5\,8)(3\,10\,7)$$

of S_{10}, written in cycle notation, and let h be the element

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 9 & 2 & 7 & 5 & 6 & 1 & 8 & 10 \end{pmatrix}$$

of S_{10}, written in two-line notation.

(c) Write g in two-line notation. **[3]**

(d) Compute $h^{-1} \circ g^{-1}$, and write it in cycle notation. Show all your working. **[9]**

End of Paper.