Queen Mary
University of London

Main Examination period 2017

MTH4104
Introduction to Algebra

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: A. Fink, S. Beheshti

Question 1. [10 marks] Let x be a real number such that $x \neq 1$. Use mathematical induction to prove that

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

for every natural number $n \geqslant 1$. **[10]**

Question 2. [13 marks]

(a) Give the definition of a **partition** of a set X. **[3]**

(b) Write down:

(i) a set X, and a relation on X which is neither symmetric nor transitive. **[2]**

(ii) a partition of $\mathbb{Z}$ in which every part has cardinality two. **[2]**

(c) Let $\{A_1, A_2, \ldots\}$ be a partition of a set X. Prove that the relation R on X defined by

$$xRy \text{ if and only if there is some } i \text{ such that } x \in A_i \text{ and } y \in A_i$$

is an equivalence relation. **[6]**

Question 3. [21 marks]

(a) Use Euclid's algorithm to find the greatest common divisor of 288 and 111. Show all your working. **[6]**

(b) Does the equation $288x + 111y = 6$ have a solution where x and y are integers? Find one if so, showing your working, or explain why not if not. **[10]**

(c) Define what it means for an element of a ring to be a **unit**. **[2]**

(d) Is $[111]_{288}$ a unit in the ring $\mathbb{Z}_{288}$? Why? **[3]**

Question 4. [14 marks] Let $\mathbb{H} = \{\alpha + \beta j : \alpha, \beta \in \mathbb{C}\}$ be the set of quaternions. Define a function $\varphi : \mathbb{H} \to \mathrm{M}_2(\mathbb{C})$ by

$$\varphi(\alpha + \beta j) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

(a) Write down the definition of multiplication for quaternions. **[2]**

(b) Prove that $\varphi(q \cdot r) = \varphi(q) \cdot \varphi(r)$ for any two quaternions $q, r \in \mathbb{H}$. **[4]**

(c) Prove that φ is an injective function. **[3]**

(d) Use parts (b) and (c) to prove that the quaternions satisfy the associative law for multiplication. You may assume that $\mathrm{M}_2(\mathbb{C})$ is a ring. **[5]**

Question 5. [14 marks]

(a) Let R be a ring. Define what it means for R to be

(i) a **commutative ring**; **[2]**

(ii) a **skewfield**. **[2]**

Give the full statement of any axioms you invoke.

(b) Let R be a ring. Prove from the axioms that $a \cdot 0 = 0$ for any $a \in R$. **[6]**

(c) Let R be a ring, and $a \in R$ an element such that $a^2 = 0$. Must it be true that $a = 0$? Justify your answer. **[4]**

Question 6. [14 marks]

(a) Let G and H be groups, with respective operations $\circ$ and $*$. Define what it means for

(i) G to be a **subgroup** of H; **[2]**

(ii) G and H to be **isomorphic**. **[2]**

(b) Prove that

$$\{a^2/b^2 : a \text{ and } b \text{ are nonzero integers}\}$$

is a subgroup of the multiplicative group $\mathbb{Q}^\times$. **[6]**

(c) Suppose that G is a nonabelian group and H is an abelian group. With reference to the definition, explain why G and H cannot be isomorphic. **[4]**

Question 7. [14 marks] Let g be the element

$$(1\ 3\ 10)(2\ 5\ 12)(4\ 6\ 7\ 11\ 9)$$

of S_{12}, written in cycle notation, and let h be the element

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 4 & 12 & 5 & 3 & 10 & 2 & 11 & 1 & 9 & 8 & 7 & 6 \end{pmatrix}$$

of S_{12}, written in two-line notation.

(a) Write g in two-line notation. [**3**]

(b) Compute $(gh)^{-1}$ and write your answer in cycle notation. [**6**]

(c) Define the **order** of an element of a group. [**2**]

(d) What is the order of h? [**3**]

End of Paper.