Queen Mary
University of London

Main Examination period 2017

MTH4103/MTH4203: Geometry I

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: Robert Johnson

Question 1. Let $\mathbf{a} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$ and $M = \begin{pmatrix} 4 & 2 & 1 \\ -1 & 6 & 0 \\ -7 & 1 & -1 \end{pmatrix}$.

(a) Determine:

(i) $\mathbf{a}+\mathbf{b}$; [**3**]

(ii) a unit vector in the direction of $-\mathbf{a}$; [**3**]

(iii) $\mathbf{a}\times\mathbf{b}$; [**3**]

(iv) $\det(M)$; [**3**]

(v) M^2. [**4**]

(b) Verify that $\mathbf{a}$ is an eigenvector of M and find the corresponding eigenvalue. [**4**]

Question 2. Consider the following system of linear equations in variables x, y, z.

$$\left.\begin{aligned} 2x - y + z &= 1 \\ -2x + 2y - z &= 1 \\ 4x + y + z &= 2 \end{aligned}\right\}$$

(a) What geometric problem is equivalent to finding the solutions to this system? [**3**]

(b) Perform the first step (dealing with the x variable) of the Gaussian elimination algorithm to this system of equations. [**5**]

(c) Is the system of equations that results from the operations in part (b) in echelon form? Give a reason for your answer. [**3**]

(d) Give an example of a system of three linear equations in x, y, z which is not in echelon form and for which the first step of the Gaussian elimination algorithm uses a different type of operation from the one followed in part (b). Indicate what step the algorithm follows in your example. [**3**]

(e) Define what it means for the $n\times n$ matrix A to be **invertible**. [**3**]

(f) Describe an alternative method for solving the original system of equations involving the inverse of a matrix. You should specify the matrix in question but you do not need to show that it is invertible or calculate its inverse. [**3**]

Question 3.

(a) State the Triangle Rule for vector addition. Your answer should express the sum of two (free) vectors without using coordinates. [**4**]

(b) Without using coordinates, prove the associative law for vector addition; that is, for any vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ we have $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$. [Hint: Use the Triangle Rule.] [**6**]

Let A, B and C be the points with position vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

respectively. Let l be the line which passes through the point A and meets the line segment BC in its midpoint.

(c) Find a vector equation for l. [**5**]

(d) Find the distance from the origin to the line l. [**5**]

Question 4.

(a) For each of the following equations, describe the geometric objects it defines giving enough detail to completely determine the object. [**10**]

(i) $2x+3y+4z=0$, (ii) $2x=3y=4z$, (iii) $\left| \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right| = 1$

(b) Determine equations for x, y (and if appropriate z) which specify the following sets. [**10**]

(i) The set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ which are fixed by the linear transformation of $\mathbb{R}^2$ corresponding to the matrix $\begin{pmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{pmatrix}$,

(ii) The set of vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$ which are mapped to $\mathbf{0}$ by the linear transformation of $\mathbb{R}^3$ corresponding to the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$,

(iii) The set of eigenvectors of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Question 5.

(a) Define what it means for a function $t : \mathbb{R}^n \to \mathbb{R}^n$ to be a **linear transformation** of $\mathbb{R}^n$. [**4**]

(b) Determine whether the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x+z \\ 0 \\ y-1 \end{pmatrix}$$

is a linear transformation of $\mathbb{R}^3$ justifying your answer.

[You may use any properties of linear maps proved in lectures provided that you state them clearly.] [**4**]

(c) Let g be a linear transformation of $\mathbb{R}^3$ and let M be the corresponding 3×3 matrix. Give an interpretation of the first column of M in terms of the linear transformation g. [**3**]

(d) Each of the following functions is a linear transformation. For each one, determine the corresponding matrix. [**9**]

(i) $a : \mathbb{R}^3 \to \mathbb{R}^3$; $\quad a\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y+z \\ x+y+z \end{pmatrix}$,

(ii) $b : \mathbb{R}^3 \to \mathbb{R}^3$; $\quad b(\mathbf{u}) = \mathbf{u} \times \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$,

(iii) $c : \mathbb{R}^2 \to \mathbb{R}^2$; where c consists of reflection of the plane in the x-axis followed by reflection of the plane in the y-axis.

End of Paper.