Queen Mary
University of London

B. Sc. Examination by course unit 2015

MTH4103: Geometry I

Duration: 2 hours

Date and time: 30 April 2015, 10.00am

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): Robert Johnson

Question 1. Let $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$.

Find:

(a) The length of the vector $2\mathbf{a} + \mathbf{b}$; [**3**]

(b) The distance from the origin to the point with position vector $2\mathbf{a} + \mathbf{b}$; [**2**]

(c) A unit vector in the direction of $\mathbf{a}$; [**3**]

(d) The cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$; [**3**]

(e) A non-zero vector orthogonal to $\mathbf{a} + \mathbf{b}$; [**3**]

(f) $\mathbf{a} \times \mathbf{b}$; [**3**]

(g) A linear transformation $t : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $t(\mathbf{b}) = \mathbf{a}$. [**3**]

Question 2.

(a) Define precisely what it means for a system of linear equations (which may contain degenerate equations) to be in *echelon form.* [**5**]

(b) Use the method of back substitution to find all solutions to the following system of linear equations in echelon form: [**5**]

$$\left.\begin{aligned} 2x + y - 2z &= 1 \\ y - z &= 0 \\ 2z &= 6 \end{aligned}\right\}$$

(c) State precisely what your answer to part (b) means regarding the intersection of a specific collection of planes in 3-space. [**4**]

Question 3.

(a) Let Π be a plane with vector equation $\mathbf{r} \cdot \mathbf{n} = d$, and let Q be a point with position vector $\mathbf{q}$. Prove that the distance from Q to Π is **[6]**

$$\frac{|\mathbf{q} \cdot \mathbf{n} - d|}{|\mathbf{n}|}.$$

(b) Find d so that the distance from the plane with equation $x - y + 2z = d$ to the origin is 1. **[5]**

Question 4.

(a) Define in terms of vectors what it means for the figure $ABCD$ to be a parallelogram. **[5]**

(b) Let A, B and C be points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively, and let D be the point such that $ABCD$ is a parallelogram. Let U be the midpoint of $\overrightarrow{AB}$, V be the midpoint of $\overrightarrow{AC}$, and W be the midpoint of $\overrightarrow{CD}$.

Show that U, V, W lie on a straight line and find an equation for that line. **[10]**

Question 5.

(a) Define what it means for a function $t : \mathbb{R}^n \to \mathbb{R}^m$ to be a *linear transformation* (also called a *linear map*). **[4]**

(b) Show that if $t : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then there exists a matrix A such that $t(\mathbf{u}) = A\mathbf{u}$ for all $\mathbf{u} \in \mathbb{R}^2$. **[6]**

(c) For each of the following functions state whether or not it is a linear transformation. For those that are, give the corresponding matrix. For those that are not, provide a justification. **[9]**

(i) $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+1 \\ y+1 \\ z+1 \end{pmatrix}$,

(ii) $g : \mathbb{R}^3 \to \mathbb{R}^3$, $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y \\ 3z \\ x \end{pmatrix}$,

(iii) the rotation of the plane $\mathbb{R}^2$ about the origin anticlockwise by angle θ.

Question 6. Let M be the following 3×3 matrix:

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & -4 & 1 \end{pmatrix}$$

(a) Calculate M^2. [**3**]

(b) Calculate $M\mathbf{v}$ when $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. [**3**]

(c) Find all of the eigenvalues of M. [**5**]

(d) For each of these eigenvalues find a corresponding eigenvector. [**5**]

(e) Choose a specific non-zero vector $\mathbf{u} \in \mathbb{R}^3$ and calculate $M^{100}\mathbf{u}$. [**5**]
[Hint: A judicious choice of $\mathbf{u}$ will help considerably.]

End of Paper.