Queen Mary
University of London

Main Examination period 2023 – January – Semester A

MTH771P / MTH771U: Foundations of Mathematical Modelling in Finance

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **3 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Only approved non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: Dr M. Phillips, Dr P. Vergel

Question 1 [25 marks].

(a) Let X_1 and X_2 be two independent continuous random variables, each with the probability density function

$$f_X(x) = \begin{cases} \sqrt{\frac{2}{\pi}}\, e^{-x^2/2} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Calculate the expectation and variance of X_1 and X_2. [**3**]

(ii) Now calculate the expectation and variance of the sum $Z = X_1 + X_2$. [**2**]

(iii) Finally, determine the probability density function of Z by evaluating the integral

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_X(z-x)\, dx.$$

(**Note**: Your solution may be expressed in terms of the cumulative distribution function of the standard normal distribution.) [**5**]

(b) Suppose that I toss a coin N times. The coin is biased, with p_h being the probability that a given toss is a head. You may assume that $N > 0$ and $0 < p_h < 1$.

Let H_n denote the number of heads in the first n tosses $(0 < n \leq N)$.

Determine the correlation of H_m and H_n, where $0 < m \leq N$ and $0 < n \leq N$. [**6**]

(c) Consider a European put option with strike price K and expiry time T. The underlying share does not pay any dividends during the lifetime of the option.

Suppose that, at some earlier time t, the market price of the option is P_t, and the market price of the share is S_t (which is strictly positive). The continuously-compounded risk-free interest rate is r.

Show that P_t must satisfy each of the following conditions, otherwise there will be arbitrage opportunities:

(i) $P_t > 0$. [**3**]

(ii) $P_t < Ke^{-r(T-t)}$. [**3**]

(iii) $P_t \geq Ke^{-r(T-t)} - S_t$. [**3**]

(**Hint**: For each case separately, identify an arbitrage opportunity if the condition does not hold.)

Question 2 [20 marks].

Instruction: Answer parts (a) and (b) of this question in the context of a general one-period market model with a finite number of assets (including a bank account or other risk-free asset) and a finite number of outcomes.

(a) Explain what is meant by the following terms:

(i) a portfolio, [**1**]

(ii) an arbitrage opportunity, [**1**]

(iii) a risk-neutral probability measure, [**1**]

(iv) a complete market. [**1**]

(b) State the First and Second Fundamental Theorems of Asset Pricing. [**4**]

Now consider the following specific one-period model, with two risky assets and three possible outcomes:

	Price at time 0	Price at time 1		
		Outcome 1	Outcome 2	Outcome 3
Asset A	35	40	46	32
Asset B	25	50	35	10

There is also a bank account paying an interest rate $R = 0.2$ per period.

(c) Verify that there is no arbitrage in this market, and that the market is complete.

(**Note**: Show all the steps of your proof. Clearly state any theorems that you use, at the points in your proof when you use them.) [**6**]

(d) Suppose that an option has payoff at time 1 given by

$$V_1 = \max(A_1 + B_1 - K,\ 0),$$

where A_1 and B_1 are the prices at time 1 of assets A and B respectively, and the strike price $K = 50$. Such an option is known as a basket option.

Determine the fair price V_0 of this option at time 0. [**2**]

(e) Suppose that I sell 500 of these options at time 0.

How many units (shares) of assets A and B would I need to buy or sell, also at time 0, in order to hedge my position? [**4**]

Question 3 [10 marks]. Consider the four-period binomial model, with market parameters $S_0 = 100$, $u = 1.5$, $d = 0.7$ and $R = 0.1$. (S_0 is the initial price of the stock, u and d denote the multiplicative jump sizes of the stock price at each time step, and R is the risk-free interest rate per period.) Denote the stock price process as $S = (S_0, S_1, S_2, S_3, S_4)$.

Consider a lookback option that has payoff at time $N = 4$ given by

$$V_4 = \max\big(\max(S_1, S_2, S_3, S_4) - K,\, 0\big),$$

where $K = 160$ is the strike price.

Calculate the fair price V_0 of this option at time 0. **[10]**

Question 4 [15 marks].

(a) (i) Give a formal definition of the Wiener process. **[4]**

(ii) Explain how geometric Brownian motion is related to the Wiener process. **[1]**

(b) A European log option is a derivative that pays

$$V_T(S_T) = \max\left(\log\left(\frac{S_T}{K}\right), 0\right)$$

at expiry time T, where S_T is the stock price at expiry and K is the strike price. In this question you will consider a particular log option that has strike price $K = 98$ and expiry time eight months from today (i.e. $T = \frac{2}{3}$ years).

The current price of the underlying stock is $S_0 = 102$, and the stock price is assumed to follow geometric Brownian motion with annualised drift $\mu = 0.15$ and volatility $\sigma = 0.24$. You may assume that the underlying stock pays no dividends during the lifetime of the option. The continuously-compounded annualised risk-free interest rate is $r = 0.05$.

(i) Estimate the current price of this log option, using the discrete-time binomial model with 8 periods. **[7]**

(ii) Compare this estimate with the exact Black-Scholes value, given by

$$V_{0,\mathrm{BS}} = e^{-rT}\sigma\sqrt{T}\Big\{d_-\Phi(d_-) + \phi(d_-)\Big\},$$

where

$$d_- = \frac{\log(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

As usual, $\Phi(x)$ and $\phi(x)$ are the cumulative distribution function and probability density function respectively of the standard normal distribution. You will need to use one value from the following table:

x	$\Phi(x)$
0.160 786	0.563 869
0.276 275	0.608 832
0.356 745	0.639 359
0.472 234	0.681 620

[3]

Question 5 [30 marks].

(a) Suppose that we have a call and a put option, both European and both on the same underlying stock, with the same strike price K and expiry time T. The underlying stock pays no dividends during the lifetime of the options. There is also a bank account with continuously-compounded interest rate r.

Derive the put-call parity formula relating the option prices (C_t and P_t) at some earlier time $t \le T$.

(**Note**: Ensure that you clearly justify each step in your proof.) [**6**]

(b) The Black-Scholes price at time $t = 0$ of a European call option is given by

$$C_0 = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

with

$$d_+ = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_- = d_+ - \sigma\sqrt{T},$$

and where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution. The other symbols have their usual meanings.

Using the put-call parity formula, derive a formula for the Black-Scholes price P_0 of the corresponding European put option, expressing your solution in the most compact form possible. [**2**]

(c) By (partially) differentiating the expression for P_0 with respect to S_0, find the formula for the delta Δ of this put option. [**6**]

(d) By differentiating once again, find the formula for the gamma Γ. [**2**]

(e) Show that Γ (when viewed as a function of S_0, with all other quantities kept fixed) takes its maximal value when S_0 takes the value

$$S_0^* = K\exp\left[-\left(r + \frac{3\sigma^2}{2}\right)T\right].$$ [**8**]

(f) Finally, find a compact formula for the theta Θ of this put option, where $\Theta = -\dfrac{\partial P_0}{\partial T}$. [**6**]

End of Paper.