Main Examination period 2022 – May/June – Semester B

MTH6158: Ring Theory

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: F. Rincón, A. Fink

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Question 1 [28 marks].

(a) Is the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = 3x$ a ring homomorphism? Justify your answer. **[4]**

(b) Consider the map $\phi : \mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ defined as $\phi(m) = [5m]_{10}$.

(i) Prove that ϕ is a ring homomorphism. **[6]**

(ii) Describe explicitly the image $\mathrm{Im}(\phi)$ and the kernel $\mathrm{Ker}(\phi)$ of ϕ. How many elements do $\mathrm{Im}(\phi)$ and $\mathrm{Ker}(\phi)$ have? **[6]**

(iii) Use the First Isomorphism Theorem to express your answer for $\mathrm{Im}(\phi)$ in part (ii) as isomorphic to a factor ring of $\mathbb{Z}$. Write down explicitly the isomorphism between these two rings. **[6]**

(c) How many ring homomorphisms are there from the ring $\mathbb{Z}$ to the ring $\mathbb{Z}/10\mathbb{Z}$? Justify your answer. **[6]**

Question 2 [24 marks]. Let R be the Boolean ring $R = \mathcal{P}(\{a, b, c\})$, with addition being symmetric difference and multiplication being intersection. Consider the ideals $I = \mathcal{P}(\{a\})$ and $J = \mathcal{P}(\{b, c\})$ of R.

(a) How many cosets of I in R are there? **[2]**

(b) Write down explicitly the partition of R into these cosets. **[4]**

(c) Explain why $I + J = R$. **[6]**

(d) Use the third isomorphism theorem to show that the factor ring R/J is isomorphic to the subring I. **[6]**

(e) Use the second isomorphism theorem to conclude that if K is an ideal of R containing J then $K = J$ or $K = R$. **[6]**

Question 3 [26 marks].

Consider the subring $S = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ of the ring $\mathbb{C}$ of complex numbers.

(a) Explain why S is an integral domain. [**4**]

(b) Is the subset $T = \{c + 2d\sqrt{-3} : c, d \in \mathbb{Z}\}$ an ideal of S? Justify your answer. [**6**]

(c) Determine all the units of S. Justify your answer. [**6**]

(d) Provide an example showing that S is **not** a unique factorisation domain. You do not need to prove anything. [**4**]

(e) Is the element $7 + \sqrt{-3} \in S$ irreducible? Justify your answer. [**6**]

Question 4 [22 marks].

Suppose R is a ring such that $a^2 = 0$ for all $a \in R$.

(a) Can the ring R be an integral domain? Explain. [**4**]

(b) Explain why **no** element of R can be a unit. [**6**]

(c) Prove that for any $a, b \in R$ we must have $a \cdot b = -(b \cdot a)$. [**6**]

(d) Give an example of such a ring R in which there exist $a, b \in R$ with $a \cdot b \neq 0$. [**6**]

End of Paper.