

Main Examination period 2022 – January – Semester A

## MTH5129: Probability & Statistics II

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final.**

**IFoA exemptions.** For actuarial students, this module counts towards IFoA actuarial exemptions. To be eligible for IFoA exemption, **you must submit your exam within the first 3 hours of the assessment period.**

Examiners: C. Beck, P. Bhuyan

In this exam,  $P(\cdot)$  denotes a probability measure defined on a space  $(\Omega, \mathcal{F})$  and  $E(\cdot)$  denotes the expectation with respect to  $P$ .

**Question 1 [30 marks].** Suppose that  $X$  and  $Y$  have a joint probability density function given by

$$f_{X,Y}(x, y) = \begin{cases} ce^{-3x-5y} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Determine the value of the normalization constant  $c$ . [5]
- (b) Find the marginal probability density function  $f_X$  and state the name of the distribution of  $X$ . [8]
- (c) Find the conditional probability density function  $f_{Y|X=x}$ . [8]
- (d) Are the random variables  $X$  and  $Y$  statistically independent? Justify your answer. [3]

Consider now a different joint probability density function for  $X$  and  $Y$ , namely

$$\tilde{f}_{X,Y}(x, y) = \begin{cases} 12ye^{-3x-2y^2} & \text{if } x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- (e) What is the probability  $P(Y^2 > 2X > 0)$ ? [6]

**Question 2 [10 marks].** Suppose that  $X_1$  and  $X_2$  have joint probability density function

$$f_{X_1, X_2}(x_1, x_2) = \begin{cases} \frac{1}{6}x_1x_2, & \text{if } 1 < x_1 < 2 \text{ and } 1 < x_2 < 3 \\ 0, & \text{otherwise.} \end{cases}$$

What is the joint probability density function  $f_{Y_1, Y_2}$  of  $Y_1 = X_1/X_2$  and  $Y_2 = X_2$ ? [10]

**Question 3 [10 marks].** Suppose that  $X$ ,  $Y$  and  $Z$  are statistically independent random variables, each of them with a  $\chi^2(2)$  distribution.

- (a) Find the moment generating function of  $U = X + 3Y + Z$ . State clearly and justify all steps taken. [7]
- (b) Calculate the expectation  $E(U)$  using the moment generating function. [3]

*Hint: You may use without proof the fact that the moment generating function of a  $\chi^2(\nu)$  random variable  $W$  is*

$$M_W(t) = \left( \frac{1}{1-2t} \right)^{\nu/2}.$$

**Question 4 [25 marks].** Transport engineers are interested to study the pattern and the risk of accidents at a junction. Suppose we observe the following frequency distribution of the number of accidents in 150 weeks.

Number of Accidents	0	1	2	3	4	5	6	7+
Observed frequency	50	30	24	30	10	5	1	0

- (a) Find an estimate (to four decimal places) of the average number of accidents in a week. [6]
- (b) Perform a goodness of fit test at the 5% significance level of the null hypothesis that the observed number of accidents follow a Poisson distribution. Hint: Use the estimate (to four decimal places) you computed in (a) and report the expected frequency and observed value of the statistic to four decimal places. [15]
- (c) What is the p-value of your test in (b)? Does the p-value indicate that there is evidence against the null hypothesis? [4]

**Question 5 [25 marks].** Twenty patients sampled at random were matched by age and BMI. One of each pair was assigned at random to a new treatment and the other to an existing treatment. Ultrasound examination of a certain tumour weight (in grams) produced the following results.

Existing Treatment	15.5	5	8.5	10.6	12.5	6.5	8.4	10.5	8	10
New Treatment	14	5.7	9	11.5	10	7	7.5	8	9.9	10.5

To answer the following questions, report the numerical computations to four decimal places.

- (a) Test whether there is a difference in the two treatments at 5% level of significance. [10]
- (b) Find a 90% confidence interval for the mean difference in the treatments. [5]
- (c) What would be the conclusion in (a) if we had wrongly ignored the pairing? [10]

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**End of Paper.**