Queen Mary
University of London

Main Examination period 2019

MTH5129: Probability & Statistics II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

Only non-programmable calculators that have been approved from the college list of non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

The New Cambridge Statistical Tables are provided.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: N. Rodosthenous, L.I. Pettit

Question 1. [14 marks]

(a) Define what it means for two random variables X and Y to be **independent**. **[4]**

(b) Suppose X and Y have a joint probability density function given by

$$f(x,y) = \begin{cases} Kxy, & \text{if} \quad 0 < x < 1, \quad 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the value of K. **[6]**

(ii) Calculate the probability $P(X < 0.5, Y < 0.75)$. **[4]**

Question 2. [13 marks] Suppose that X and Y are two random variables and their joint density function $f_{X,Y}$ is given by

$$f_{X,Y}(x,y) = \begin{cases} 8xy, & \text{if} \quad 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal probability density function $f_X(x)$ of the random variable X. **[8]**

(b) Find the conditional probability density function $f_{Y|X=x}(y)$. **[5]**

Question 3. [12 marks] Suppose that a random walk on a line is starting from n, where $M \leqslant n \leqslant N$ and $M, n, N \in \mathbb{Z}$. The probability of a jump to the right is $p \in (0,1)$ and the probability of a jump to the left is $q = 1 - p$. The walk stops once it reaches either M or N. Let r_n be the probability that the walk starting from n reaches N before M.

(a) State the equations for r_n, where $M \leqslant n \leqslant N$ and explain how r_M and r_N are obtained. **[7]**

(b) The probabilities r_n (equivalently, the solution to the equations from question part (a)) are given by

$$r_n(M,N) = \begin{cases} \dfrac{\left(\frac{q}{p}\right)^n - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M}, & \text{if } p \neq q \\ \frac{n-M}{N-M}, & \text{if } p = q. \end{cases}$$

Suppose now that a particle starts from position 3 and has probabilities of jumps $p = 2/3$ and $q = 1/3$. We also suppose that the particle is free to go as far right as it wishes. What is the probability that it eventually reaches 0? **[5]**

Question 4. [6 marks]

Prove Markov's inequality, which is stated below.
Theorem (Markov's Inequality) Suppose that X is a non-negative random variable. Then for any number $\delta > 0$

$$P(X \geqslant \delta) \leqslant \frac{E(X)}{\delta}.$$ **[6]**

Question 5. [15 marks]

A random sample of size n is assumed to be normally distributed with unknown variance σ^2. The sample variance is S^2. The distribution of the statistic $W = \frac{(n-1)S^2}{\sigma^2}$ is chi-squared with $n-1$ degrees of freedom (χ^2_{n-1}).

(a) Derive the form of a $100(1-\alpha)\%$ confidence interval for σ^2. **[4]**

(b) If $n = 9$ and the observed sample variance is $s^2 = 16.3$ find a 99% confidence interval for σ^2. **[4]**

(c) Carry out a two sided test of the hypothesis that $\sigma^2 = 10.0$ using a significance level of 5%. **[7]**

Question 6. [20 marks]

In medieval times, before the advent of printing, scribes copied books by hand, and errors naturally occurred in the process of copying. In studying a book and one particular copy of it, a random sample of 100 pages was examined and the numbers of errors per page were recorded. The data are summarised in the following table.

Number of errors per page	0	1	2	3	4	5	6	7	8
Observed Frequency	10	32	24	22	9	2	0	0	1

(a) Explain why it might be reasonable to assume that the number of errors per page would follow a Poisson distribution. **[3]**

(b) Calculate for these data the sample mean of the number of errors per page. **[2]**

(c) Carry out a goodness-of-fit test at the 5% significance level to examine the null hypothesis that the number of errors per page has a Poisson distribution. **[15]**

Question 7. [20 marks] Eight patients who suffered from severe insomnia took part in a study to determine the effects of two sedatives. Each patient took sedative A for a two week period and the average number of hours sleep were recorded for each patient. This procedure was then repeated for sedative B. The results were as follows.

Patient	1	2	3	4	5	6	7	8
Sedative A	2.1	2.9	5.4	3.8	3.1	4.1	2.4	2.9
Sedative B	1.6	2.0	5.2	4.0	3.3	3.2	1.8	2.3

(a) Test the hypothesis that the effects of the two sedatives are the same using a significance level of 5%. Make clear what assumptions you are making. **[10]**

(b) Find a 95% confidence interval for the difference in average amount of sleep under the two sedatives. **[5]**

(c) Comment on the design of this trial by noting one good feature and one improvement which could be made. **[5]**

End of Paper.