Queen Mary
University of London

Main Examination period 2023 – May/June – Semester B

MTH5114: Linear Programming and Games

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

You should attempt ALL questions. Marks available are shown next to the questions.

You are allowed to bring **three A4 sheets of paper** as notes for the exam.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: V. Patel, J. Ward

Question 1 [20 marks]. A steel production company has three different processes in place to produce varying amounts of two different grades of steel. The amounts of iron, coal, and limestone (in tons) needed to run each process for one hour is detailed in the table below. In addition the table gives the amounts (in tons) of each grade of steel produced from running each process for one hour.

Process	Iron Required	Coal Required	Limestone Required	Low Grade Steel Produced	High Grade Steel Produced
1	14	3	8	11	0
2	17	5	10	0	9
3	40	13	28	12	18

Each ton of high grade steel can be sold for £1500 and each ton of low grade steel can be sold for £1000. The company currently has 250 tons of iron, 200 tons of coal and 300 tons of limestone available.

(a) The company wants to know how to generate as much revenue as possible by producing and selling steel using the processes described above (you may assume that there is no limit on the number of hours each process can be run). Give a linear program that models this problem. You do not need to solve this program. [**9**]

(b) Suppose now that, any of the 200 tons of coal that is not used, is sold at a price of £400 per ton, which then contributes to the revenue of the company. Everything else remains unchanged. Again, the company wants to know how to generate as much revenue as possible. How should the linear program from the previous part be changed to model the new situation? You do not need to solve this program. [**3**]

(c) Rewrite the following linear program in **standard inequality form**. You do not need to write out each intermediate linear program, but briefly mention which steps you take to arrive at your final answer. [You do **not** need to write your answer in matrix form.]

$$\begin{aligned} \text{minimize} \quad & 7x_1 - x_2 + 5x_3 - 10x_4 \\ \text{subject to} \quad & -11x_1 - 12x_2 - x_3 + x_4 \geq 1, \\ & x_1 - 4x_2 - 8x_3 = 12, \\ & 2x_1 + 6x_2 + 3x_4 \leq 7, \\ & x_1, x_2 \geq 0, \\ & x_3 \leq 0, \\ & x_4 \text{ unrestricted.} \end{aligned}$$

[**8**]

Question 2 [30 marks].

Consider a linear program in standard equation form:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

(a) Prove (without assuming any results from lectures) that if $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions of the linear program, then every convex combination of $\mathbf{x}$ and $\mathbf{y}$ is also an optimal solution of the linear program. [**10**]

(b) Now suppose $\mathbf{x}$ and $\mathbf{y}$ are both optimal solutions to the linear program above where $\mathbf{x} \neq \mathbf{y}$. Show that $\mathbf{c}$ is orthogonal to the line that passes through $\mathbf{x}$ and $\mathbf{y}$. [**5**]

(c) Give the dual of the following linear program:

$$\begin{aligned} \text{maximize} \quad & x_1 - 2x_2 + 3x_3 \\ \text{subject to} \quad & x_1 + x_2 - 2x_3 \leq 1, \\ & 2x_1 - x_2 - 3x_3 \leq 4, \\ & x_1 + x_2 + 5x_3 \leq 2, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

[**6**]

(d) Use complementary slackness to decide if $x_1 = 9/7$, $x_2 = 0$, $x_3 = 1/7$ is an optimal solution to the linear program above. Fully justify your answer. [**9**]

Question 3 [25 marks].

(a) Solve the following linear program using the simplex algorithm. Show all your working and indicate which row operations you are doing at each stage. Clearly write down the final optimal solution. [Hint: the objective value of the optimal solution is 25/2.]

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 + x_3 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 10, \\ & x_1 + 2x_2 - 2x_3 \leq 20, \\ & x_2 + 2x_3 \leq 5, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

[15]

(b) Consider the following linear program.

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 6x_2 - x_3 \\ \text{subject to} \quad & x_1 + 2x_2 + x_3 \leq 4, \\ & x_1 - x_2 \geq 3, \\ & -3x_1 + x_2 - 2x_3 = 2, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(i) Apply phase 1 of the 2-phase simplex algorithm up to the point where the initial tableau is brought into a valid form. **[7]**

(ii) What can you conclude about the linear program above from your answer to part (b)(i). **[3]**

Question 4 [25 marks].

(a) Rosemary and Colin play a game in which each of them writes down a number from $\{1, 2, 3\}$ without showing their number to the other player. They then reveal their numbers to each other.

Suppose the sum of the numbers is s. If $s \leq 4$, then Rosemary wins an amount s, while Colin wins an amount $s - 2$. If $s \geq 5$, then Colin wins an amount $8 - s$, while Rosemary wins an amount 0.

(i) Write down the payoff matrix for this game, where, as usual, Rosemary's payoff is given first. [**4**]

(ii) Find Collin's best response to each of Rosemary's (pure) strategies. [**2**]

(iii) Find all of the pure Nash equilibria for this game. Briefly explain how you obtain your answer. [**5**]

(b) Consider the 2-player zero-sum game with the following payoff matrix (which is given, as usual, from the perspective of the row player).

	c_1	c_2.
r_1	6	-6
r_2	3	9

(i) Write a linear program that finds the optimal mixed strategy for the row player (i.e. the mixed strategy with the best security level). You do not have to solve this linear program. [**6**]

(ii) Consider the mixed strategy $\mathbf{x}$ for the row player and $\mathbf{y}$ for the column player given by $\mathbf{x}^\mathsf{T} = (1/3, 2/3)$ and $\mathbf{y}^\mathsf{T} = (5/6, 1/6)$. Show that this pair of strategies is a mixed Nash equilibrium for this game. [**8**]

End of Paper.