Queen Mary
University of London

Main Examination period 2019

MTH5114: Linear Programming and Games

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: J. Ward, M. Jerrum

Question 1. [18 marks]

(a) Rewrite the following linear program in **standard inequality form**:

$$\begin{aligned}
\text{minimise} \quad & -3x_1 + 6x_2 + 9x_3 + 12x_4 \\
\text{subject to} \quad & x_1 + x_2 - x_4 \geq 3, \\
& x_1 + x_2 - 3x_4 \leq 5, \\
& 3x_1 + 2x_2 + x_3 = 9, \\
& x_1, x_2 \geq 0, \\
& x_4 \leq 0, \\
& x_3 \text{ unrestricted}
\end{aligned}$$

[**6**]

(b) Now, consider an arbitrary linear program in **standard equation form**:

$$\begin{aligned}
\text{maximise} \quad & \mathbf{c}^{\mathsf{T}}\mathbf{x} \\
\text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}$$

with n variables and m constraints.

(i) Explain what it means for $\mathbf{x} \in \mathbb{R}^n$ to be a **basic feasible solution** of this linear program. [**4**]

(ii) Let $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{z} \in \mathbb{R}^n$ be two optimal solutions to this linear program and let $\lambda \in (0, 1)$ be a constant. Show that $\mathbf{x} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{z}$ must also be an optimal solution to this program. [**8**]

Question 2. [**18 marks**] A clothing company produces two different types of shirts: casual shirts, and dress shirts. Each shirt requires some amount of fabric (measured in metres), buttons, and labour (measured in hours). The company has production lines for producing each type of shirt separately. Additionally, the company has determined that it can waste less fabric if it cuts parts for both types of shirts out of the same length of cloth. It has developed a third production line for this method, which costs slightly more labour.
The total amount of resources required and the number of shirts produced by using each production line for one day is shown below:

Production Line	Fabric Required	Buttons Required	Labour Required	Casual Shirts Produced	Dress Shirts Produced
1	140	210	85	70	0
2	125	400	75	0	50
3	200	500	180	60	40

(a) Suppose that the company has only 1500 metres of fabric, 3500 buttons, and 1800 hours of labour available. It can sell an unlimited number of casual shirts for £45 each and an unlimited number of dress shirts for £65 each. It wants to know the most revenue it can generate using only these available resources. Write a linear program that models this problem (you do not need to solve this program). [**10**]

(b) Now suppose that the company has the same amount of resources available as in part (b), and can still sell any number of casual shirts at a price of £45. However, it has determined that its retailers can sell only up to 250 dress shirts at a price of £65, and then up to 300 additional dress shirts at a discounted sale price of £39. Again, the company wants to know the most revenue it can generate using only its available resources. Write a linear program that models this optimisation problem. [**8**]

Question 3. [26 marks]

(a) Suppose that at some point when carrying out the standard simplex algorithm on a linear program in standard equation form, you have the following current tableau:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	2	0	0	2	4
x_3	0	-6	1	0	2	2
x_4	0	8	0	1	0	3
	0	-3	0	0	2	-8

(i) What basic feasible solution does this tableau correspond to? **[4]**

(ii) What is the objective value of this basic feasible solution? **[2]**

(iii) What is the entering variable and what is the leaving variable for the next pivot operation? **[4]**

(iv) Give the next tableau produced by the simplex algorithm. **[4]**

(b) Suppose that while running the initial phase of the 2-phase simplex algorithm on some linear program, you have the following current tableau (where a_1 and a_2 are artificial variables):

	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	-5	3	0	1	0	0	-3	15
a_1	-6	3	0	0	-1	1	-1	6
x_3	3	-2	1	0	0	0	1	1
$-w$	-6	3	0	0	-1	0	-2	6
$-z$	-16	15	0	0	0	0	-6	-6

(i) Finish running the first phase of the algorithm, giving the tableau for each step. **[8]**

(ii) Say whether or not the linear program is feasible. If it is feasible, give the initial tableau for the second phase. If it is not feasible, say why not. **[4]**

Question 4. [20 marks]

(a) Give the dual of the following linear program:

$$\begin{aligned} \text{maximise} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 \\ \text{subject to} \quad & x_1 - 2x_2 + 6x_3 + 7x_4 \le 9, \\ & x_1 + 5x_2 + 10x_3 + 15x_4 = 25, \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

[**8**]

(b) State the strong duality theorem for linear programs. [**4**]

(c) Consider the following linear program:

$$\begin{aligned} \text{maximise} \quad & 6x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & 3x_1 + 3x_2 + 2x_3 \le 9, \\ & x_1 + 2x_2 + x_3 \le 4, \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Show that $x_1 = 3$, $x_2 = 0$, $x_3 = 0$ is an optimal solution of this linear program. [**8**]

Question 5. [18 marks]

(a) Consider a 2-player game with the following payoff matrix. Here, the row player's strategies are r_1, r_2, and r_3 and the column player's strategies are c_1, c_2, and c_3. As usual, the row player's payoff is given first and the column player's payoff is given second in each cell.

	c_1	c_2	c_3
r_1	$(1, 2)$	$(0, 1)$	$(2, 3)$
r_2	$(2, 0)$	$(1, 2)$	$(2, 1)$
r_3	$(0, 1)$	$(0, 1)$	$(1, 2)$

Does this game possess a pure Nash equilibrium? If so, list **all** pure Nash equilibria for this game. [**4**]

(b) What does it mean for a 2-player game to be a **zero-sum game**? [**3**]

Now, consider the following zero-sum game. The row player and the column player each simultaneously hold up 1 or 2 fingers. If the sum of the number of fingers both players hold up is even, the row player wins. If the sum is odd, the column player wins. The losing player pays the winning player an amount equal to the number of fingers the winner held up.

(c) Give the payoff matrix for this game from the perspective of the row player. [**5**]

(d) Formulate a linear program that gives the row player's optimal mixed strategy for this game (you do not need to solve this program). [**6**]

End of Paper.