Queen Mary
University of London

Main Examination period 2019

MTH5113: Introduction to Differential Geometry

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: A. Shao, S. Majid

There is a compendium of definitions and formulae in the appendix, which you are free to use without comment.

Question 1. [22 marks] Let C be the curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid (x+3)^2 + 4(y-2)^2 = 16\}$$

and consider the following parametrisation of C:

$$\gamma : \mathbb{R} \to \mathbb{R}^2, \qquad \gamma(t) = (-3 + 4\cos t, 2 + 2\sin t).$$

(a) Find the curvature of C at the point $(-3, 0)$. [6]

(b) Sketch the image of γ, and indicate the point $(-3, 0)$ on your sketch. [5]

(c) At which points of C does its curvature achieve its **maximum** value? Justify your answer(s) computationally. [4]

(d) Compute the curve integral

$$\int_C \mathbf{F} \cdot \mathbf{ds},$$

where C has the **clockwise** orientation, and where $\mathbf{F}$ is the vector field given by

$$\mathbf{F}(x, y) = (-y,\, x)_{(x,y)}, \qquad (x, y) \in \mathbb{R}^2.$$ [7]

Question 2. [14 marks]

(a) Compute the tangent line at $t = 0$ to the parametric trefoil knot:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \qquad \gamma(t) = (\sin t + 2\sin(2t), \cos t - 2\cos(2t), -\sin(3t)).$$ [5]

(b) Determine whether the following parametric curve is regular:

$$\alpha : \mathbb{R} \to \mathbb{R}^2, \qquad \alpha(t) = ((t-1)^3, (t-1)^2).$$

Justify your answer. [5]

(c) Give a parametrisation of the curve,

$$Q = \{(x, y) \in \mathbb{R}^2 \mid x^4 + (y+2)^4 = 1\},$$

that passes through the point $(0, -1)$. **Be sure to specify its domain.** [4]

Question 3. [23 marks] Let S denote the surface of revolution given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^4 = y^2 + z^2,\ 0 < x < 2\}$$

and consider the following parametrisation of S:

$$\sigma : (0, 2) \times \mathbb{R} \to \mathbb{R}^3, \qquad \sigma(u, v) = (u, u^2 \cos v, u^2 \sin v).$$

(a) Compute the tangent plane to S at the point $(1, 0, 1)$. **[5]**

(b) Sketch the image of σ. On your sketch, draw (i) a path obtained by holding v constant and varying u, and (ii) a path obtained by holding u constant and varying v. **[6]**

(c) Find another parametrisation of S that generates the **opposite orientation** to the one generated by σ. **Be sure to specify its domain.** **[4]**

(d) Compute the surface integral

$$\iint_S H \, dA,$$

where H is the function

$$H : \mathbb{R}^3 \to \mathbb{R}, \qquad H(x, y, z) = \sqrt{1 + 4x^2}.$$ **[8]**

Question 4. [14 marks]

(a) Let $\mathbf{f}$ denote the following vector-valued function:

$$\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{f}(x, y) = (xy^2, x^2 y).$$

Find the directional derivative of $\mathbf{f}$ at the point $(1, 1)$ and in the direction $(-1, 2)$. **[5]**

(b) Explain (informally) why the surface integral of a real-valued function over a Möbius band is well-defined, but the surface integral of a vector field over the same Möbius band is **not** well-defined. **[4]**

(c) Show that the following set is a surface:

$$Z = \{(x, y, z) \in \mathbb{R}^3 \mid x = y^3 + z^4\}.$$ **[5]**

Question 5. [15 marks] Using the method of Lagrange multipliers, find the maximum value of the function

$$g : \mathbb{R}^2 \to \mathbb{R}, \qquad g(x, y) = x^2 + y^2,$$

subject to the constraint

$$(x - 1)^2 + (y + 1)^2 = 1.$$

Also, find all the points at which this maximum value is achieved. [**15**]

Question 6. [12 marks]

(a) Let C be the circle centred at the origin and having radius 2,

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\},$$

with the **anticlockwise** orientation. Use **Green's theorem** to compute

$$\int_C \mathbf{F} \cdot \mathbf{ds},$$

where $\mathbf{F}$ is the vector field on $\mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = (xe^{x^2} \ln(1 + x^2) - 3y, 3x + y^{18} \sinh y \cos y^2)_{(x,y)}.$$

(You may use that the area of the inside of a circle with radius R is πR^2.) [**8**]

(b) Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (x^2 z + e^y,\ z^3 y^3 x^4,\ 1 + x^3 y^2)_{(x,y,z)}.$$

Compute the divergence of $\mathbf{F}$ at each point $(x, y, z) \in \mathbb{R}^3$. [**4**]

End of Paper – An appendix of 3 pages follows.

Partial list of definitions, theorems, and formulas

- *Tangent vector*: $\mathbf{v_p}$ ($\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$)—arrow starting at $\mathbf{p}$, pointing in direction $\mathbf{v}$.
- *Tangent space* of $\mathbb{R}^n$ at $\mathbf{p} \in \mathbb{R}^n$: $T_{\mathbf{p}}\mathbb{R}^n = \{\mathbf{v_p} \mid \mathbf{v} \in \mathbb{R}^n\}$.
- Operations on tangent vectors ($\mathbf{p}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$; $c \in \mathbb{R}$):
$$\mathbf{v_p} + \mathbf{w_p} = (\mathbf{v} + \mathbf{w})_{\mathbf{p}}, \qquad c \cdot \mathbf{v_p} = (c \cdot \mathbf{v})_{\mathbf{p}}, \qquad |\mathbf{v_p}| = |\mathbf{v}|,$$
$$\mathbf{v_p} \cdot \mathbf{w_p} = (\mathbf{v} \cdot \mathbf{w})_{\mathbf{p}}, \qquad \mathbf{v_p} \times \mathbf{w_p} = (\mathbf{v} \times \mathbf{w})_{\mathbf{p}}.$$
- *Vector field* on $A \subseteq \mathbb{R}^n$: function $\mathbf{F}$ mapping each $\mathbf{p} \in A$ to $\mathbf{F}(\mathbf{p}) \in T_{\mathbf{p}}\mathbb{R}^n$.
- **Thm.** The *directional derivative* of a smooth function $\mathbf{g} : U \to \mathbb{R}^n$ ($U \subseteq \mathbb{R}^m$) at the point $\mathbf{p} \in U$ and in the direction $\mathbf{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$ satisfies
$$\mathbf{dg}(\mathbf{v_p}) = v_1 \cdot \partial_1 \mathbf{g}(\mathbf{p}) + \cdots + v_m \cdot \partial_m \mathbf{g}(\mathbf{p}).$$
- *Gradient* of $f : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^m$): vector field ∇f on U, with
$$\nabla f(\mathbf{p}) = (\partial_1 f(\mathbf{p}), \dots, \partial_m f(\mathbf{p}))_{\mathbf{p}}.$$
- For a vector field $\mathbf{F}$ on $U \subseteq \mathbb{R}^m$, with $\mathbf{F}(\mathbf{p}) = (F_1(\mathbf{p}), \dots, F_m(\mathbf{p}))_{\mathbf{p}}$:
  - *Divergence* of $\mathbf{F}$ at $\mathbf{p} \in U$:
$$(\nabla \cdot \mathbf{F})(\mathbf{p}) = \partial_1 F_1(\mathbf{p}) + \cdots + \partial_m F_m(\mathbf{p}).$$
  - $n = 3$: *curl* of $\mathbf{F}$ at $\mathbf{p} \in U$:
$$(\nabla \times \mathbf{F})(\mathbf{p}) = (\partial_2 F_3(\mathbf{p}) - \partial_3 F_2(\mathbf{p}),\ \partial_3 F_1(\mathbf{p}) - \partial_1 F_3(\mathbf{p}),\ \partial_1 F_2(\mathbf{p}) - \partial_2 F_1(\mathbf{p}))_{\mathbf{p}}.$$
- *Parametric curve*: smooth $\gamma : I \to \mathbb{R}^n$ (I: open interval).
  - γ is *regular* iff $|\gamma'(t)| \neq 0$ for every $t \in I$.
- Informally, a *curve* $C \subseteq \mathbb{R}^n$: (1) is described using regular parametric curves, (2) is independent of parametrisation, and (3) is not self-intersecting.
- *Parametrisation* of curve C: any regular parametric curve $\gamma : I \to C$.
- **Thm.** If $g : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^2$) is smooth, with ∇g nonvanishing on U, then any nonempty level set $C = \{(x, y) \in U \mid g(x, y) = c\}$ ($c \in \mathbb{R}$) is a curve. Also:
  - $\nabla g(\mathbf{p})$ is perpendicular to $T_{\mathbf{p}}C$ for each $\mathbf{p} \in C$.
  - *Lagrange multipliers*: If $f : U \to \mathbb{R}$ is smooth, and if $\mathbf{p} \in C$ is a maximum or minimum of f on C, then $\nabla f(\mathbf{p}) = \lambda \cdot \nabla g(\mathbf{p})$ for some $\lambda \in \mathbb{R}$.
- **Thm.** The graph $\{(t, f(t)) \mid t \in I\}$ of any smooth $f : I \to \mathbb{R}$ is a curve.
- *Tangent line* of:
  - Regular parametric curve γ at t: $T_\gamma(t) = \{s \cdot \gamma'(t)_{\gamma(t)} \mid s \in \mathbb{R}\}$.
  - Curve C at $\mathbf{p}$: $T_{\mathbf{p}}C = T_\gamma(t)$ (γ: parametrisation of C, with $\gamma(t) = \mathbf{p}$).
- *Orientation* of curve C: smoothly varying choice of unit $\mathbf{T_p} \in T_{\mathbf{p}}C$ for all $\mathbf{p} \in C$.
  - A parametrisation γ generates an orientation via values $|\gamma'(t)|^{-1}\gamma'(t)_{\gamma(t)}$.
- *Oriented curve*: roughly, a curve with a choice of orientation.
- *Curvature* of:
  - **Thm.** Regular parametric curve $\gamma : I \to \mathbb{R}^2$ at t:
$$\kappa_\gamma(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{|\gamma'(t)|^3}, \qquad \gamma(t) = (x(t), y(t)).$$

  - **Thm.** Regular parametric curve $\gamma : I \to \mathbb{R}^3$ at t:
$$\kappa_\gamma(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}.$$
  - Curve C at $\mathbf{p}$: $\kappa_C(\mathbf{p}) = \kappa_\gamma(t)$ (γ: parametrisation of C, with $\gamma(t) = \mathbf{p}$).
- *Curve integral* of real-valued function F, and *arc length*:
  - Over parametric curve $\gamma : (a, b) \to \mathbb{R}^n$:
$$\int_\gamma F\, ds = \int_a^b F(\gamma(t))|\gamma'(t)|dt, \qquad L(\gamma) = \int_\gamma 1\, ds.$$
  - Over curve C (γ: injective parametrisation of "almost all of" C):
$$\int_C F\, ds = \int_\gamma F\, ds, \qquad L(C) = L(\gamma).$$
- *Curve integral* of vector field $\mathbf{F}$:
  - Over parametric curve $\gamma : (a, b) \to \mathbb{R}^n$:
$$\int_\gamma \mathbf{F} \cdot ds = \int_a^b \left[\mathbf{F}(\gamma(t)) \cdot \gamma'(t)_{\gamma(t)}\right] dt.$$
  - Over oriented curve C (γ: as before, and with matching orientation):
$$\int_C \mathbf{F} \cdot ds = \int_\gamma \mathbf{F} \cdot ds.$$
- *Parametric surface*: smooth $\sigma : U \to \mathbb{R}^n$ ($U \subseteq \mathbb{R}^2$: open, connected).
  - σ is *regular* iff $\partial_1\sigma$, $\partial_2\sigma$ are everywhere linearly independent.
  - **Thm.** $n = 3$: σ is regular iff $|\partial_1\sigma \times \partial_2\sigma| \neq 0$ everywhere.
- Informally, a *surface* $S \subseteq \mathbb{R}^n$: (1) is described using regular parametric surfaces, (2) is independent of parametrisation, and (3) is not self-intersecting.
- *Parametrisation* of surface S: any regular parametric surface $\sigma : U \to S$.
- **Thm.** If $g : U \to \mathbb{R}$ ($U \subseteq \mathbb{R}^3$) is smooth, and if ∇g is nonvanishing on U, then any nonempty level set $S = \{(x, y, z) \in U \mid g(x, y, z) = c\}$ ($c \in \mathbb{R}$) is a surface. Also:
  - $\nabla g(\mathbf{p})$ is perpendicular to $T_\mathbf{p}S$ for each $\mathbf{p} \in S$.
  - *Lagrange multipliers*: If $f : U \to \mathbb{R}$ is smooth, and if $\mathbf{p} \in S$ is a maximum or minimum of f on S, then $\nabla f(\mathbf{p}) = \lambda \cdot \nabla g(\mathbf{p})$ for some $\lambda \in \mathbb{R}$.
- **Thm.** The graph $\{(u, v, f(u, v)) \mid (u, v) \in U\}$ of any smooth $f : U \to \mathbb{R}$ is a surface.
- *Tangent plane* of:
  - Parametric surface σ at (u, v):
$$T_\sigma(u, v) = \{a \cdot \partial_1\sigma(u, v)_{\sigma(u,v)} + b \cdot \partial_2\sigma(u, v)_{\sigma(u,v)} \mid a, b \in \mathbb{R}\}.$$
  - Surface S at $\mathbf{p} \in S$: $T_\mathbf{p}S = T_\sigma(u, v)$ (σ: parametrisation of S; $\sigma(u, v) = \mathbf{p}$).
- $\mathbf{N_p}$ is a *unit normal* to a surface S at $\mathbf{p}$ iff $\mathbf{N_p}$ is normal to $T_\mathbf{p}S$ and $|\mathbf{N_p}| = 1$.
  - **Thm.** The unit normals to S at $\sigma(u, v)$ (σ: parametrisation of S) are:
$$\mathbf{N}_\sigma^\pm(u, v) = \pm\left[\frac{\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)}{|\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)|}\right]_{\sigma(u,v)}.$$
- *Orientation* of surface S: smoothly varying choice of unit normal $\mathbf{N_p}$ at all $\mathbf{p} \in S$.
  - S is *orientable* iff an orientation of S exists.

  - A parametrisation σ generates an orientation via the values $\mathbf{N}^+_\sigma(u, v)$.
- *Oriented surface*: roughly, a surface with a choice of orientation.
- *Surface integral* of real-valued function F, and *surface area*:
  - Over regular parametric surface $\sigma : U \to \mathbb{R}^3$:

$$\iint_\sigma F\, dA = \iint_U F(\sigma(u, v))|\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)|\, du dv, \qquad \mathcal{A}(\sigma) = \iint_\sigma 1\, dA.$$

  - Over surface S (σ: injective parametrisation of "almost all of" S):

$$\iint_S F\, dA = \iint_\sigma F\, dA, \qquad \mathcal{A}(S) = \mathcal{A}(\sigma).$$

- *Surface integral* of vector field $\mathbf{F}$:
  - Over regular parametric surface $\sigma : U \to \mathbb{R}^3$:

$$\iint_\sigma \mathbf{F} \cdot \mathbf{dA} = \iint_U \left\{ \mathbf{F}(\sigma(u, v)) \cdot [\partial_1\sigma(u, v) \times \partial_2\sigma(u, v)]_{\sigma(u,v)} \right\} du dv.$$

  - Over oriented surface S (σ: as before, and with matching orientation):

$$\int_S \mathbf{F} \cdot \mathbf{dA} = \iint_\sigma \mathbf{F} \cdot \mathbf{dA}.$$

- **Thm.** *Green's theorem*: ($D \subseteq \mathbb{R}^2$: open, bounded, boundary given by curves $C_1, \dots, C_k$.) For a smooth vector field $\mathbf{F}$, with $\mathbf{F}(\mathbf{p}) = (F_1(\mathbf{p}), F_2(\mathbf{p}))_{\mathbf{p}}$,

$$\int_{C_1} \mathbf{F} \cdot \mathbf{ds} + \cdots + \int_{C_k} \mathbf{F} \cdot \mathbf{ds} = \iint_D [\partial_1 F_2(x, y) - \partial_2 F_1(x, y)]\, dx dy.$$

($C_1, \dots, C_k$ given the "positive"—left from outward normal—orientation.)
- **Thm.** *Stokes' theorem*: ($S \subseteq \mathbb{R}^3$: oriented surface, boundary given by curves $C_1, \dots, C_k$.) For a smooth vector field $\mathbf{F}$,

$$\int_{C_1} \mathbf{F} \cdot \mathbf{ds} + \cdots + \int_{C_k} \mathbf{F} \cdot \mathbf{ds} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{dA}.$$

($C_1, \dots, C_k$ given the "positive" orientation from the chosen side of S.)
- **Thm.** *Divergence theorem*: ($V \subseteq \mathbb{R}^3$: open and bounded, with boundary given by surfaces $S_1, \dots, S_k$.) For a smooth vector field $\mathbf{F}$,

$$\int_{S_1} \mathbf{F} \cdot \mathbf{dA} + \cdots + \int_{S_k} \mathbf{F} \cdot \mathbf{dA} = \iiint_V (\nabla \cdot \mathbf{F})\, dx dy dz,$$

($S_1, \dots, S_k$ given the outward-facing orientation from V.)

End of Appendix.