Queen Mary
University of London

Main Examination period 2020 – January – Semester A

MTH5112: Linear Algebra 1

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: I. Tomašić, B. Jackson

Question 1 [14 marks].

(a) Let V be a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. When do we say that

vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ **span** V?

(Give a precise definition.) [**3**]

(b) Consider vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \text{ and } \mathbf{v}_4 = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} \text{ in } \mathbb{R}^3.$$

(i) Do vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span $\mathbb{R}^3$? [**5**]

(ii) Do vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ span $\mathbb{R}^3$? [**3**]

(iii) Are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ linearly independent? [**3**]

Justify your answer in each case, and state precisely any theorems you use.

Question 2 [14 marks].

(a) Let V be a vector space and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. When do we say that

vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent**?

(Give a precise definition.) [**3**]

(b) Consider vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 4 \\ 7 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 8 \\ 0 \end{pmatrix} \text{ and } \mathbf{v}_3 = \begin{pmatrix} 3 \\ 6 \\ 9 \\ 1 \end{pmatrix} \text{ in } \mathbb{R}^4.$$

(i) Are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ linearly independent? [**5**]

(ii) Are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ linearly independent? [**3**]

(iii) Do vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span $\mathbb{R}^4$? [**3**]

Justify your answer in each case, and state precisely any theorems you use.

Question 3 [10 marks]. Let P_2 denote the vector space of polynomials of degree at most 2. Consider the subset

$$H = \{\mathbf{p} \in P_2 : \mathbf{p}(1) = \mathbf{p}(0)\}.$$

(a) Show that H is a subspace of P_2. **[5]**

(b) Find a basis for H and determine $\dim(H)$. **[5]**

Question 4 [18 marks]. Let P_2 denote the vector space of polynomials of degree at most 2, and let

$$D : P_2 \to P_2$$

be the transformation that sends a polynomial $\mathbf{p}(t) = at^2 + bt + c$ in P_2 to its derivative $\mathbf{p}'(t) = 2at + b$, that is,

$$D(\mathbf{p}) = \mathbf{p}'.$$

(a) Prove that D is a linear transformation. **[4]**

(b) Find a basis for the kernel $\ker(D)$ of the linear transformation D and compute its **nullity**. **[4]**

(c) Find a basis for the image $\mathrm{im}(D)$ of the linear transformation D and compute its **rank**. **[4]**

(d) Verify that the Rank-Nullity Theorem holds for the linear transformation D. **[3]**

(e) Find the matrix representation of D in the standard basis $(1, t, t^2)$ of P_2. **[3]**

Question 5 [16 marks].

(a) Define the **norm** $\|\mathbf{u}\|$ of a vector $\mathbf{u} \in \mathbb{R}^n$. **[3]**

(b) When are vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ considered **orthogonal**? **[3]**

(c) When do we say that a set $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ of vectors in $\mathbb{R}^n$ is **orthonormal**? **[4]**

(d) Prove the following statement.

> If the set $\{\mathbf{u}, \mathbf{v}\}$ is orthonormal, then the vectors $\mathbf{u}, \mathbf{v}$ are linearly independent.

[6]

Question 6 [20 marks]. Let

$$A = \begin{pmatrix} -1 & -2 & 2 \\ 4 & 3 & -4 \\ 0 & -2 & 1 \end{pmatrix}.$$

(a) Show that $\mathbf{v} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of A and find the corresponding eigenvalue. [**4**]

(b) Find the characteristic polynomial of A and factorise it. **Hint:** the answer to (a) may be useful. [**5**]

(c) Determine all eigenvalues of A and find bases for the corresponding eigenspaces. [**7**]

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [**4**]

Question 7 [8 marks]. Consider the least squares problem $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 5 \end{pmatrix}.$$

(a) Write down the corresponding normal equations. [**4**]

(b) Determine the set of least squares solutions to the problem. [**4**]

End of Paper.