Queen Mary
University of London

Main Examination period 2018

MTH5112: Linear Algebra I

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: T. Popiel and C. Busuioc

Question 1. [16 marks]

(a) Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & x_4 & = & 6 \\ -x_1 & & & - & x_3 & & & = & -1 \\ 2x_1 & + & 2x_2 & - & 2x_3 & + & 3x_4 & = & 14 \end{array}$$

(i) Write down the augmented matrix of the system. **[2]**

(ii) Put the augmented matrix into **reduced** row echelon form (RREF), indicating which elementary row operation you have used at each step. **[5]**

(iii) State which of the variables are leading variables and which are free variables, and write down the solution set of the system. **[3]**

(b) Use the Gauss–Jordan algorithm to find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

indicating which elementary row operation you have used at each step. **[6]**

Question 2. [17 marks]

(a) Suppose that A is an invertible matrix. Prove that the system of linear equations $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. **[5]**

(b) Consider the matrices

$$B = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} -1 & 1 \\ 3 & 0 \\ 1 & 2 \end{pmatrix}.$$

(i) Compute the matrix $B^T - 2B$. **[2]**

(ii) For each of the matrix products

$$BC \quad \text{and} \quad CB,$$

either compute the product, or explain why it is not defined. **[4]**

(c) Give examples of the following:

(i) a symmetric 3×3 matrix; **[3]**

(ii) an upper triangular 3×3 matrix. **[3]**

Question 3. [18 marks]

(a) Consider the matrix

$$A = \begin{pmatrix} 7 & -5 & 1 & 4 \\ 0 & 0 & 4 & 0 \\ 3 & -1 & 2 & 2 \\ 0 & 1 & -6 & 0 \end{pmatrix}.$$

(i) Compute the determinant of A. **[4]**

(ii) Using your answer to part (i), explain whether A is invertible or not. (Do **not** attempt to compute the inverse.) **[2]**

(b) Suppose that B is a square matrix with $\det(B) = 5$. Compute the following:

(i) $\det(B^3)$; **[2]**

(ii) $\det(B^{-1})$; **[2]**

(iii) $\det(B^T)$. **[2]**

(c) Suppose that C is a 4×4 matrix with $\det(C) = 3$. Compute the determinants of the following matrices:

(i) the matrix obtained by swapping the first and second rows of C; **[2]**

(ii) the matrix obtained by multiplying the fourth row of C by -6; **[2]**

(iii) the matrix obtained by subtracting the third row of C from the first row. **[2]**

Question 4. [15 marks]

(a) Prove that

$$H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}$$

is a subspace of $\mathbb{R}^3$. **[5]**

(b) Write down a set of five linearly independent vectors in $\mathbb{R}^4$, or explain why it is impossible to do so. **[4]**

(c) Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 5 & 2 & 0 \\ 0 & 1 & -4 & -2 & 1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 & -6 \end{pmatrix}.$$

(i) Write down a basis for the row space of A, and determine the rank of A. **[3]**

(ii) Using your answer to part (i), determine the nullity of A. **[3]**

Question 5. [16 marks] Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find the eigenvalues of A. **[4]**

(b) For each of the eigenvalues of A, find a basis for the corresponding eigenspace. **[6]**

(c) Using your answer to part (b), find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$, or explain why this is impossible. **[6]**

Question 6. [18 marks]

(a) Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

(i) Show that $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. **[6]**

(ii) Write down the transition matrix from $\mathcal{B}$ to the standard basis of $\mathbb{R}^3$. **[4]**

(iii) Find the best approximation to the vector

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

by vectors in the subspace $H = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{R}^3$. **[4]**

(b) Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and suppose that the system of linear equations

$$A\mathbf{x} = \mathbf{b}$$

has no solutions. What does it mean to say that a vector $\mathbf{x} \in \mathbb{R}^n$ is a **least squares solution** of such a system? **[4]**

End of Paper.