MTH751U / MTH751P / MTHM751: Processes on Networks

Duration: 3 hours

Date and time: 12th May 2016, 14:30–17:30

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Examiner(s): G. Bianconi

Question 1 (35 marks).

Robustness of uncorrelated networks to targeted attack of the high degree nodes.
Consider an uncorrelated random network with degree distribution $P(k)$.

- We initially damage a fraction f of nodes with highest degree.
- We indicate with $k_c(f)$ the highest degree of the nodes that are not initially damaged.
- We indicate by S the probability that a node is in the giant component.
- We indicate by S' the probability that a link reaches a non damaged node of degree $k \leq k_c(f)$ that is in the giant component.
- The brackets $\langle \ldots \rangle$ indicate the average over the degree distribution $P(k)$.

a) Express f as a function of k_c and of the degree distribution $P(k)$. **[4]**

b) Show that S' satisfies the equation

$$S' = \sum_k \frac{kP(k)}{\langle k \rangle} \theta(k_c(f) - k) \left[1 - (1 - S')^{k-1}\right], \tag{1}$$

where $\theta(x) = 1$ if $x \geq 0$ otherwise $\theta(x) = 0$. **[5]**

c) Show that S satisfies the equation

$$S = \sum_k P(k)\theta(k_c(f) - k) \left[1 - (1 - S')^k\right]. \tag{2}$$

[5]

d) Show that in order to have a giant component in the network, i.e. $S > 0$ we must have

$$\frac{\langle k^2 \theta(k_c(f) - k)\rangle - \langle k\theta(k_c(f) - k)\rangle}{\langle k \rangle} > 1. \tag{3}$$

[5]

e) Given a scale-free network with power-law degree distribution $P(k) = Ck^{-\gamma}$ with $k \in [1, \sqrt{N}]$ and $\gamma \in (2, 3)$, calculate $\langle k\ \theta(k_c(f) - k)\rangle$ and $\langle k^2\ \theta(k_c(f) - k)\rangle$ in the mean-field, continuous approximation. **[8]**

f) Given the network of point e) and a finite $f > 0$, evaluate $k_c(f)$ in the mean-field, continuous approximation using the expression found in point a). For every finite f is $\langle k^2\ \theta(k_c(f) - k)\rangle$ calculated in point e) finite or infinite? **[8]**

Question 2 (40 marks).

The Ising model on a network.

In the mean-field approximation of the Ising model the average local magnetization $\langle s_i \rangle$ of the node spin s_i of node i in a network with adjacency matrix a_{ij} satisfies the equation

$$\langle s_i \rangle = \tanh\left(\beta J \sum_j a_{ij} \langle s_j \rangle + \beta h\right), \tag{4}$$

where β is the inverse temperature, J the coupling constant and h the external magnetic field.

a) Show that in the mean-field annealed approximation the average magnetization $\langle s_k \rangle$ of a node of degree k, satisfies

$$\langle s_k \rangle = \tanh\left(\beta J k \Theta + \beta h\right), \tag{5}$$

where

$$\Theta = \sum_k \frac{k}{\langle k \rangle} P(k) \tanh\left(\beta J k \Theta + \beta h\right). \tag{6}$$

[10]

b) Show that for $h \to 0$ there is a phase transition as a function of the temperature and that the critical temperature in the annealed network approximation is given by

$$T_c = J \frac{\langle k^2 \rangle}{\langle k \rangle}. \tag{7}$$

[10]

c) Show that for $\langle k^2 \rangle / \langle k \rangle \to \infty$, the critical temperature found at point c) is a first order approximation of the exact result found by the cavity method

$$T_c = 2J \left[-\ln\left(1 - 2\frac{\langle k \rangle}{\langle k^2 \rangle}\right)\right]^{-1}. \tag{8}$$

[6]

d) Consider an uncorrelated scale-free networks with power-law degree distribution $P(k) = Ck^{-\gamma}$, $\gamma = 3$ and $k \in [1, \sqrt{N}]$. Evaluate $\langle k \rangle$ and $\langle k^2 \rangle$ in the continuous approximation. **[8]**

e) Consider the network of point d). What is the value of the critical temperature T_c given by Eq. (7) in the limit $N \to \infty$? What is the value of the critical temperature T_c given by Eq. (8) in the limit $N \to \infty$? **[6]**

Question 3 (25marks).

The SIR model on complex networks.
Consider the SIR model on a complex network, where β is the rate at which a susceptible individual in contact with an infected individual becomes infected, and μ is the rate at which an infected individual becomes removed.

a) Show that the probability density function $P(\tau)$ of times τ required for an infected individual to become removed is given by

$$P(\tau) = \mu e^{-\mu\tau}. \tag{9}$$

[**8**]

b) The transmissibility T is given by the probability that an infected node transmits the infection to a nearest neighbour in the susceptible state. Show that the transmissibility T is given by

$$T = 1 - \int d\tau P(\tau) e^{-\beta\tau} = \frac{\lambda}{1+\lambda} \tag{10}$$

where $\lambda = \beta/\mu$. [**8**]

c) Map the SIR model on a network to the percolation process on the same network, by identifying the transmissibility T of the SIR model with the probability p that a random node is not damaged in the percolation transition. Show that the value λ_c is given by

$$\lambda_c = \frac{\langle k \rangle}{\langle k^2 \rangle - 2\langle k \rangle}. \tag{11}$$

[**6**]

d) Which is this the epidemic threshold in a regular network of degree distribution $P(k) = \delta_{k,3}$? [**3**]

End of Paper—An appendix of 2 pages follows.