Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH751U: Processes on networks

Duration: 3 hours

Date and time: 21st May 2015, 10:00-13:00

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You should attempt ALL questions. Marks awarded are shown next to the questions.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): G. Bianconi

Question 1 (35 marks).

Percolation of uncorrelated networks.

Consider an uncorrelated random network with degree distribution $P(k)$ and average degree $\langle k \rangle$, where the nodes are randomly damaged.
Let S be the probability that a node is in the giant component.
Let S' be the probability that following a link we reach a node that is in the giant component.
Let p denote the probability that a node is not initially damaged.
a) Show that S' satisfies the equation

$$S' = p\left[1 - \sum_{k=0}^{\infty} \frac{kP(k)}{\langle k \rangle}(1-S')^{k-1}\right]. \tag{1}$$

[10]

b) Show that S satisfies the equation

$$S = p\left[1 - \sum_{k=0}^{\infty} P(k)(1-S')^{k}\right]. \tag{2}$$

[10]

c) Show that in order to have a giant component in the network, i.e. $S > 0$, we must have

$$p\frac{\langle k(k-1)\rangle}{\langle k \rangle} > 1. \tag{3}$$

[10]

d) Starting from the condition $p\frac{\langle k(k-1)\rangle}{\langle k \rangle} > 1$ for having a giant component, show that we can recover the Molloy-Reed condition for the existence of a giant component in an uncorrelated network that is not damaged, i.e.

$$\frac{\langle k^2 \rangle}{\langle k \rangle} > 2. \tag{4}$$

[5]

Question 2 (25 marks).

Robustness of uncorrelated networks with given degree distribution $P(k)$.
a) Calculate the generating function

$$G(x) = \sum_k P(k)x^k \tag{5}$$

for a Poisson random network with average degree $\langle k \rangle = c$ and degree distribution

$$P(k) = \frac{1}{k!}c^k e^{-c}. \tag{6}$$

[5]

b) Using the properties of generating functions, calculate $\langle k(k-1)\rangle$ for a Poisson random network with average degree $\langle k \rangle = c$ and degree distribution given by equation (6). Therefore show that the percolation threshold of these networks is

$$p_c = \frac{1}{\langle k \rangle} = \frac{1}{c}. \tag{7}$$

[8]

c) Consider the uncorrelated scale-free networks with degree distribution $P(k) = Ck^{-\gamma}$ with power -law exponent $\gamma \leq 3$, and structural cutoff $K = \sqrt{\langle k \rangle N}$. Calculate $\langle k^2 \rangle$ in the large N limit in the continuous approximation for the degrees of the nodes. Show that these scale-free networks have percolation threshold

$$p_c \to 0 \tag{8}$$

as $N \to \infty$. **[12]**

Hint:The percolation threshold p_c of a network is fixed by the equations

$$p_c \frac{\langle k(k-1)\rangle}{\langle k \rangle} = 1. \tag{9}$$

Question 3 (40 marks).

Susceptible-Infected-Susceptible Model
Consider the Susceptible-Infected-Susceptible (SIS) model defined on a given network with N nodes.
Let λ be the probability that a susceptible node in contact with an infected node gets the infection.
The mean-field dynamic equation for the probability ρ_i that a node $i = 1, 2, \ldots, N$ is infected is given by

$$\dot{\rho}_i = -\rho_i + \lambda(1 - \rho_i) \sum_{j=1}^{N} a_{ij}\rho_j \qquad (10)$$

where a_{ij} indicates the (i, j) matrix element of the adjacency matrix $\mathbf{a}$ of the network.
a) Find the stationary solution of (10) **[5]**
b) Decompose ρ_i into the eigenvector $f_i(\Lambda)$ of the adjacency matrix $\mathbf{a}$, corresponding to the eigenvalue Λ, i.e.

$$\rho_i = \sum_{\Lambda} c_\Lambda f_i(\Lambda), \qquad (11)$$

where the eigenvectors $f_i(\Lambda)$ satisfy $\sum_{i=1}^{N} f_i(\Lambda) f_i(\Lambda') = \delta(\Lambda, \Lambda')$ and where $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ otherwise. Find the general expression for determining the coefficients c_Λ from the vector ρ_i in a given network. **[5]**
c) Using the results obtained in (a) and (b), find the expression that the coefficients c_Λ satisfy at stationarity. **[15]**
d) Using the result in (c) find the epidemic threshold of the SIS model in the mean-field approximation, assuming that close to the transition we have $\rho_i \simeq c_{\Lambda_1} f_i(\Lambda_1) \ll 1$ where Λ_1 is the maximum eigenvalue of the adjacency matrix $\mathbf{a}$. **[15]**

End of Paper.