Queen Mary University of London

Main Examination period 2018

MTH745P/U: Further Topics in Algebra (Fields and Galois Theory)

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: J. N. Bray and S. Majid

Question 1. [22 marks] Let F and K be fields, with $F \leqslant K$.

(a) Define the **degree** $[K : F]$ of the field extension $K : F$. **[2]**

(b) State and prove the **Short Tower Law** for (finite) field extensions. **[10]**

(c) Write down the degrees of the following field extensions. We use ω to denote a primitive cube root of unity; thus you can take $\omega = \frac{1}{2}(-1+\sqrt{-3})$.

(i) $\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}$;
(ii) $\mathbb{Q}(\sqrt[5]{11}, \sqrt{3}) : \mathbb{Q}$;
(iii) $\mathbb{Q}(\sqrt[3]{7}, \omega\sqrt[3]{7}) : \mathbb{Q}$. **[3]**

(d) Define what it means to say that F is the **prime subfield** of K. Prove that if this is the case then $F \cong \mathbb{F}_p(= \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z})$ or $F \cong \mathbb{Q}$. **[7]**

Question 2. [32 marks] Let F and K be fields, with $F \leqslant K$.

(a) Define the notion of a **Euclidean domain**. **[4]**

(b) Indicate briefly why the polynomial ring $F[x]$ is Euclidean. **[2]**

(c) Let $f(x) \in F[x]$, and let $\lambda \in F$. Prove that $f(x)$ is divisible by $x-\lambda$ (in $F[x]$) if and only if $f(\lambda) = 0$. **[4]**

(d) Let $F \leqslant K$ be fields. Let $f(x), g(x) \in F[x]$, and let $h(x)$ be a g.c.d. of $f(x)$ and $g(x)$ in $F[x]$. Prove that $h(x)$ is also a g.c.d. of $f(x)$ and $g(x)$ in $K[x]$. **[6]**

(e) Let $f(x) \in \mathbb{Z}[x]$ be the product of two non-constant polynomials in $\mathbb{Q}[x]$. Prove that it is the product of two non-constant polynomials in $\mathbb{Z}[x]$. **[8]**

(f) State **Eisenstein's Irreducibility Criterion** for integer polynomials. **[4]**

(g) Prove that x^3-4x+2 and x^3-x-1 are irreducible over $\mathbb{Q}$. **[4]**

Question 3. [22 marks] In this question, $K : F$ is a field extension and $f(x) \in F[x]$.

(a) Define what it means for K to be a **splitting field** for $f(x)$ over F. **[4]**

(b) Prove that if K is a splitting field for $f(x)$ over F then $[K : F]$ is finite. **[4]**

(c) Define what it means for $K : F$ to be **normal**. **[4]**

(d) Prove that if $K : F$ is finite and normal then K is a splitting field over F for some $f(x) \in F[x]$. **[6]**

(e) Give examples (one of each, without proof) of finite extensions of $\mathbb{Q}$ that are

(i) normal; **[2]**

(ii) not normal. **[2]**

Question 4. [24 marks]

(a) State the **Fundamental Theorem of Galois Theory**. **[8]**

(b) Let L be a splitting field over $\mathbb{Q}$ for $x^4 - 7$. Compute the Galois groups $G = \mathrm{Gal}(L : \mathbb{Q})$, of L over $\mathbb{Q}$, and $\mathrm{Gal}(L : \mathbb{Q}(\sqrt{7}))$. (You can take L to be the subfield of $\mathbb{C}$ with this property.) **[8]**

(c) Choose **two** subgroups of $G = \mathrm{Gal}(L : \mathbb{Q})$ other than G, the trivial subgroup and any subgroup having fixed field $\mathbb{Q}(\sqrt{7})$. For **each** of your chosen subgroups H of G, give the fixed field of H, and state whether $\mathrm{Fix}(H) : \mathbb{Q}$ is a normal extension. **[8]**

End of Paper.