Queen Mary
University of London

MTH745U / MTH745P: Further topics in algebra—Fields and Galois Theory

Duration: 3 hours

Date and time: 5 May 2016, 1430h–1730h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): R. A. Wilson

Question 1. Suppose that K and L are fields, and $K \subseteq L$.

(a) Show that L is a vector space over K. [**4**]

(b) What does it mean to say that K is the **prime field** of L? Show that, if this is the case, then K is isomorphic to $\mathbb{Z}_p (= \mathbb{Z}/p\mathbb{Z})$ or $\mathbb{Q}$. [**6**]

(c) Define the **degree** $[L : K]$ of L over K. [**2**]

(d) Suppose M is a field containing L, and finite-dimensional over K. Prove that $[M : K] = [M : L][L : K]$. [**5**]

(e) Write down bases for $\mathbb{Q}(\sqrt[3]{2}, \sqrt{5})$ over

(i) $\mathbb{Q}$;
(ii) $\mathbb{Q}(\sqrt{5})$;
(iii) $\mathbb{Q}(\sqrt[3]{2})$.

[**3**]

Question 2. Suppose $L : K$ is a field extension, and $\alpha \in L$.

(a) What does it mean for α to be **algebraic over** K? [**2**]

(b) If α is algebraic over K, define the **minimal polynomial** $m(x)$ for α over K. [**4**]

(c) Prove that $m(x)$ is irreducible over K. [**4**]

(d) Prove that if $f(x) \in K[x]$ satisfies $f(\alpha) = 0$, then $m(x)$ divides $f(x)$. [**4**]

(e) Let $\alpha = (1 + i)/\sqrt{2} \in \mathbb{C}$.

(i) Determine the minimal polynomial for α over $\mathbb{R}$;
(ii) Determine the minimal polynomial for α over $\mathbb{Q}$.

[**6**]

Question 3. Let $L : K$ be a field extension, and let $f(x) \in K[x]$.

(a) What does it mean to say that L is a **splitting field** for $f(x)$ over K? [**4**]

(b) Prove that if $\deg(f(x)) = n$ then there is a splitting field M for $f(x)$ over K, with $[M : K] \leq n!$. [**8**]

(c) What does it mean to say that $L : K$ is **normal**? [**4**]

(d) Give, with brief justification, one example each of finite extensions of $\mathbb{Q}$ which are

(i) normal; [**2**]
(ii) not normal. [**2**]

Question 4. Let p be a prime number, and n a positive integer. Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ denote the field of order p.

(a) Prove that every field of order p^n is a splitting field for $X^{p^n} - X$ over $\mathbb{F}_p$. [You are not required to prove the existence of such a field.] [**4**]

(b) List the monic irreducible polynomials of degree 2 over $\mathbb{F}_3$. Briefly explain why your list is complete. [**6**]

(c) Hence, or otherwise, factorize $X^9 + 2X$ into irreducibles over $\mathbb{F}_3$. [**4**]

(d) Explain briefly a construction of a field K of order 9. Find an element $\alpha \in K$ which is a generator for the multiplicative group of non-zero elements of K. What is the minimal polynomial for α over $\mathbb{F}_3$? [**6**]

Question 5. (a) State the Fundamental Theorem of Galois Theory. [**6**]

(b) Compute the Galois group G of $L : \mathbb{Q}$, where L is a splitting field for $X^4 - 3$ over $\mathbb{Q}$. [**6**]

(c) Choose **two** subgroups of $G = \mathrm{Gal}(L : \mathbb{Q})$, other than G and the trivial subgroup. For **each** of your chosen subgroups H of G, give the fixed field of H, and state whether or not $\mathrm{Fix}(H) : \mathbb{Q}$ is a normal extension. [**8**]

End of Paper.