Queen Mary
University of London

Main Examination period 2017

MTH744U / MTH744P: Dynamical Systems

Duration: 3 hours

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Examiners: R. Klages, W. Just

Question 1. [30 marks] One-dimensional systems

a) Consider the differential equation

$$\dot{x} = 1 - \exp(-x) \quad .$$

Compute all fixed points of this equation. For each fixed point compute its stability using a linear stability analysis. **[3]**

b) Sketch the phase portrait and draw a diagram in which you show qualitatively how the solutions $x(t)$ of this differential equation depend on time t. The latter diagram should display all the fixed point solutions and their stability properties. **[4]**

c) Compute a potential for this differential equation and sketch the potential in a diagram. **[5]**

d) Solve the above differential equation for the initial value $x(0) = \ln 2$. **[6]**

e) Consider the following two functions:

$$\text{(i)} \;\; x(t) = \sin(1+t) \quad , \quad \text{(ii)} \;\; x(t) = \sqrt{2\exp(2t) - 1} \quad .$$

State in each case whether the respective function $x(t)$ is the solution of an (autonomous) differential equation of the form $\dot{x} = f(x)$. Justify your answer. **[6]**

f) If the respective function $x(t)$ in (e) is a solution of a differential equation $\dot{x} = f(x)$, derive the corresponding equation by calculating an expression for the right hand side $f(x)$. **[6]**

Question 2. [34 marks] Bifurcations

Consider the differential equation

$$\dot{x} = rx - x^2 + x^4 \quad ,$$

which depends on a real parameter r.

a) Compute the range of parameter values for which the trivial fixed point $x_* = 0$ is linearly stable and the range of parameter values for which it is linearly unstable. If you find bifurcations of the trivial fixed point, compute the parameter value of the bifurcation point and state with a reason the type of the bifurcation. **[8]**

b) Compute the parameter values $r = r_*$ and the points $x = x_*$ in phase space where nontrivial fixed points, $x_* \neq 0$, undergo a saddle node bifurcation. **[8]**

c) For each saddle node bifurcation computed in part b) decide whether the pair of fixed points is generated for $r < r_*$ or $r > r_*$. **[10]**

d) Using the results from parts a) - c), or otherwise, sketch the bifurcation diagram of the differential equation. Your diagram should indicate the stability of each fixed point. The diagram should also contain phase portraits for parameter values where phase portraits qualitatively differ. **[8]**

Question 3. [36 marks] Two-dimensional systems

Consider the system of two differential equations

$$\dot{x} = y \quad , \quad \dot{y} = -x + x^3 - y \quad .$$

a) Compute the fixed points of this two dimensional dynamical system. For each fixed point perform a linear stability analysis and classify the type of fixed point. [**6**]

b) Is this system conservative? Is it a gradient system? Is it reversible? State reasons and compute, if appropriate, the potential. [**12**]

c) Sketch the flow in the phase plane in a small neighbourhood of each fixed point. [**4**]

d) Construct a nullcline diagram for this system as follows:

(i) Compute the nullclines of the system of differential equations.

(ii) Sketch the nullclines in the phase plane.

(iii) The nullclines partition the phase plane into different regions. For each region, and on each nullcline, indicate the direction of the flow. [**8**]

e) Using the results from parts c) - d), or otherwise, sketch the full phase portrait of the two dimensional system. The phase portrait should be consistent with the diagram produced in part d). If the system has a stable fixed point then shade its basin of attraction. [**6**]

End of Paper.