Queen Mary
University of London

Main Examination period 2017

MTH743N / MTH743P / MTH743U: Complex Systems

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: V. Latora, R. Klages

Question 1. [34 marks]
Consider the map $f : [0,1] \to [0,1]$ defined as:

$$f(x) = \begin{cases} 2(x+1/6) & \text{if} \quad 0 \leq x \leq 1/3 \\ -3(x-2/3) & \text{if} \quad 1/3 < x < 2/3 \\ x-2/3 & \text{if} \quad 2/3 \leq x \leq 1 \end{cases} .$$

(a) Sketch the graph of the map. Show that the map is piecewise linear. **[6]**

(b) Show that the partition $\{I_0, I_1, I_2\}$ with $I_0 = [0,1/3]$, $I_1 = [1/3,2/3]$, $I_2 = [2/3,1]$ is a Markov partition, and find the associated topological transition matrix A. **[7]**

(c) Show that the map f is expanding. State all admissible periodic symbol sequences of period $p = 1$ and $p = 2$. **[7]**

(d) Determine, through the topological transition matrix A, the number N_p of periodic points of period p of the map f for $p = 1$, 2, 3. Compute the asymptotic behaviour of N_p for large p. **[7]**

(e) Write down the transfer matrix of the map f. Evaluate the invariant density and the Lyapunov exponent of the map. **[7]**

Question 2. [31 marks]
Consider the map $f : [0,1] \to [0,1]$ defined as: $f(x) = 4x(1-x)$.

(a) Sketch the graph of the map. Show that the map is a Markov map with $\{I_0, I_1\}, I_0 = [0,1/2], I_1 = [1/2,1]$ as a Markov partition, and write down the associated topological transition matrix. **[7]**

(b) Find the two invertible branches of the map and write down the two inverse functions. **[6]**

(c) Write down the Frobenius-Perron equation. **[6]**

(d) Show that the invariant density of the map is given by

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

i.e., show that ρ is a (normalised) solution of the Frobenius-Perron equation. **[5]**

(e) Assuming ergodicity, calculate the Lyapunov exponent:

$$\Lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|f'(x_k)|$$

where $x_k = f^{(k)}(x_0)$ denotes a (typical) orbit of the logistic map. **[7]**
(Simplify the resulting integral by adopting the change of variable $x = \sin^2 t$, and use the result: $\int_0^{\pi/2} \ln|\cos(2t))|dt = -\frac{\pi}{2}\ln 2$).

Question 3. [35 marks] Consider a network of $N = 4$ coupled dynamical systems whose topology is described by the adjacency matrix:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and whose equations read:

$$\dot{x}_i(t) = 4x_i(t) - 4x_i^2(t) + \sigma \sum_{j=1}^{4} G_{ij} x_j(t)$$

where $x_i(t)$ denotes the state at node i, $\sigma \in \mathbb{R}$ denotes the coupling strength, and G_{ij} is the Laplacian of the network.

(a) Write down a formula for the function $f(x)$ governing the local dynamics at a node, and for the function $h(x)$ determining the type of coupling. [**4**]

(b) Compute the average node degree, the degree distribution, the average distance between nodes, and the diameter of the graph. [**9**]

(c) Determine the time-independent synchronised states, and for each of them compute the master stability function. [**10**]

(d) Find the eigenvalues of the Laplacian. [**7**]

(e) For each synchronised state find the values of the coupling strength σ such that the state is transversely stable. [**5**]

End of Paper.