Main Examination period 2017

MTH742U / MTH742P: Advanced Combinatorics

Duration: 3 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: Heng Guo

In any question, you may use earlier results in later parts. For example, you may use part (c) in part (f), but not part (d) in part (c).

You may assume the Cauchy-Schwarz inequality in any question.

Question 1.

(a) State Mantel's theorem, namely the upper bound on the number of edges in any triangle-free graphs on n vertices. [**4**]

(b) Give a graph on n vertices which achieves the upper bound in Mantel's theorem. [**3**]

(c) Prove the upper bound in Mantel's theorem. [**9**]

(d) Prove that if a graph is bipartite, then it does not contain any odd cycle. [**4**]

(e) What is the maximum number of edges in a graph on n vertices without any cycle of odd length? Show that your upper bound can be achieved. [**5**]

Question 2.

(a) Let H be a finite graph with at least one edge. Define the **Turán number** $ex(n, H)$ of the graph H. [**3**]

(b) Show that the following limit exists

$$\lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}.$$ [**6**]

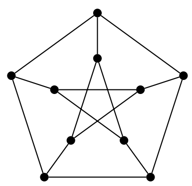
(c) Define the **Turán density** $\pi(H)$ of a finite graph H. [**3**]

(d) Define the **Chromatic number** $\chi(H)$ of a finite graph H. [**3**]

(e) State a formula relating $\pi(H)$ and $\chi(H)$. [**3**]

(f) Calculate $\pi(K_r)$ for every $r \geqslant 2$. (K_r is the complete graph on r vertices.) [**3**]

(g) Calculate $\pi(H)$ where H is the following graph:



[**4**]

Question 3.

(a) Let $G = (V, E)$ be a simple graph. Prove the handshaking lemma, namely that

$$\sum_{v \in V} d(v) = 2|E|,$$

where $d(v)$ is the degree of a vertex v. [**3**]

(b) Let $N(G)$ be the number of paths of length 2 in G. Show that

$$N(G) = \sum_{v \in V} \binom{d(v)}{2}.$$ [**4**]

(c) Let G be a bipartite graph with n vertices on each side. Let C_4 be the cycle of length 4. Show that if G is C_4-free, then $N(G) \leqslant 2\binom{n}{2}$. [**5**]

(d) With the same assumption as part (c), show that

$$|E| \leqslant \frac{n}{2}\left(\sqrt{4n-3}+1\right).$$ [**8**]

(e) With the same assumption as part (c), show that if $|E| \geqslant 2n$, then

$$N(G) \geqslant 2n.$$ [**5**]

Question 4.

(a) Define an **independent set** of a graph G. [**3**]

(b) Let G be a graph with n vertices. Let $\alpha(G)$ be the maximum size of an independent set in G.
Show that if the maximum degree of G is $d \geqslant 3$, then $\alpha(G) \geqslant \frac{n}{d+1}$. [**5**]

(c) Using the alteration method, prove that if the average degree of G is $d \geqslant 3$, then $\alpha(G) \geqslant \frac{n}{2d}$. [**8**]

(d) Let $G = (V, E)$ and $d(v)$ be the degree of v. Prove that

$$\alpha(G) \geqslant \sum_{v \in V} \frac{1}{d(v)+1}.$$ [**9**]

Question 5.

(a) Let A_1, A_2, ..., A_n be events in a probability space. Define what it means for D to be a **dependency graph** for these events. [**3**]

(b) State the (symmetric version) **Lovász Local Lemma**. [**3**]

(c) Let H be a k-uniform hypergraph with maximum degree $d \geqslant 2$. Use the local lemma to show that, if $\mathrm{e}dk \leqslant 2^{k-1}$, then H is 2-colourable. [**7**]

(d) An orientation σ of G assigns a direction to every edge in G. A vertex is a sink if all adjacent edges point toward it.

Use the local lemma to show that if G is a d-regular graph where $d \geqslant 4$, then G contains an orientation without any sink. [**7**]

(e) Let G be a connected simple graph with at least one edge. Show that G has an orientation without any sink if and only if G is not a tree. [**5**]

End of Paper.