Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH742U: Advanced Combinatorics

Duration: 3 hours

Date and time: 19 May 2015, 2:30 pm

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best FOUR questions answered will be counted.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Exam papers must not be removed from the examination room.

Examiner(s): David Ellis

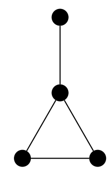
Question 1. (a) State Mantel's theorem on the maximum number of edges in a triangle-free graph. [**3**]

(b) For each positive integer n, describe the n-vertex triangle-free graphs with the maximum possible number of edges. [**3**]

(c) Write down all the different *unlabelled* triangle-free graphs with 4 vertices. [**4**]

(d) Give a proof of the upper bound in Mantel's theorem. [**8**]

(e) Let F be the 4-edge graph pictured below. For each integer $n \geq 2$, determine the F-free, n-vertex graphs with the maximum possible number of edges. Justify your answer. (You can assume any of the results in parts (a)-(d).)



[**7**]

Question 2. (a) Let F be a finite graph with at least one edge. Define the *Turán number* $\mathrm{ex}(n, F)$ of the graph F. [**3**]

(b) Show that the sequence of real numbers

$$\left(\frac{\mathrm{ex}(n,F)}{\binom{n}{2}} \right)_{n \geq 2}$$

is non-increasing. [**5**]

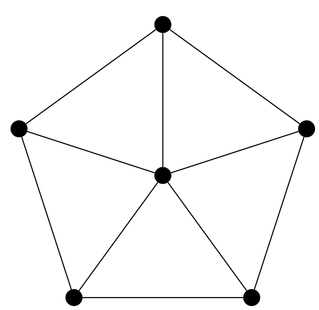
(c) Define the *Turán density* $\pi(F)$ of the graph F, as a limit, and say briefly why this limit exists. [**4**]

(d) Write down a formula for $\pi(K_s)$, valid for each integer $s \geq 2$. [**2**]

(e) Define the *chromatic number* $\chi(F)$ of the graph F. [**3**]

(f) State a formula for $\pi(F)$ in terms of $\chi(F)$. [**3**]

(g) Calculate $\pi(H)$, where H is the following graph.



[**5**]

Question 3. (a) Let G be a finite graph. Prove that G is bipartite if and only if G contains no odd cycle. [**8**]

(b) Show that if G is a finite graph with maximum degree Δ, then the 'greedy colouring algorithm' produces a proper colouring of the vertices of G with at most $\Delta + 1$ colours. [**6**]

(c) Deduce that a graph with n vertices and maximum degree Δ has an independent set of size at least

$$\frac{n}{\Delta + 1}.$$

[**4**]

(d) For each positive integer Δ, give an example of a finite graph G with maximum degree Δ and with $\chi(G) = \Delta + 1$. [**3**]

(e) State Brooks' theorem on the chromatic number of a graph. [**4**]

Question 4. (a) Let G be a graph with n vertices. Explain why

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

[**3**]

(b) Let N denote the number of paths of length 2 in G. Explain why

$$N = \sum_{v \in V(G)} \binom{d(v)}{2}.$$

[**4**]

(c) Suppose G contains no cycle of length 4. Explain why $N \leq \binom{n}{2}$. [**4**]

(d) Use parts (a), (b) and (c) to show that if G contains no cycle of length 4, then

$$e(G) \leq \tfrac{n}{4}(\sqrt{4n-3} + 1).$$

(You may assume the Cauchy-Schwarz inequality.) [**9**]

(e) Show that if $e(G) \geq \frac{1}{2}\binom{n}{2}$, then

$$N \geq \frac{n(n-1)(n-3)}{8}.$$

[**5**]

Question 5. Recall that a *tree* is defined to be a connected, acyclic graph.

(a) Let T be a tree, and let v be a vertex of T. Define what it means for v to be a *leaf* of T. **[2]**

For the rest of this question, let T be a tree with n vertices, where n is an integer greater than 1.

(b) Prove that T has at least two leaves. **[7]**

(c) Prove that $e(T) = n - 1$. **[8]**

(d) Write down the value of $\chi(T)$. **[4]**

(e) Calculate the number of (labelled) trees with vertex-set $\{1, 2, 3, 4\}$. **[4]**

Question 6. (a) Let $G_{n,p}$ denote the Erdős-Renyi random graph. Let $X = e(G_{n,p})$. Give a formula for $\mathbb{E}[X]$. **[3]**

(b) Let Y denote the number of copies of $K_{3,3}$ in $G_{n,p}$. Show that

$$\mathbb{E}[Y] = \frac{1}{2}\binom{n}{3}\binom{n-3}{3}p^9.$$

(You may assume any result in the course.) **[5]**

(c) Now let $p = n^{-2/3}$. Using part (b), show that $\mathbb{E}[Y] < \frac{1}{72}$. Use Markov's inequality to deduce an upper bound on $\text{Prob}\{Y \geq 1\}$. **[3]**

(d) Recall that if Z is a binomial random variable with $Z \sim \text{Bin}(N, p)$, and $0 \leq \delta \leq 1$, then

$$\text{Prob}\{Z \leq (1-\delta)pN\} < e^{-\delta^2 pN/2}.$$

Use this to show that

$$\text{Prob}\left\{X \leq \tfrac{1}{2}\binom{n}{2}n^{-2/3}\right\} < \tfrac{1}{2},$$

provided n is sufficiently large. (Here, you are free to take n to be as large as you want — for example, you can take $n \geq 100$.) **[4]**

(e) Using parts (c) and (d), show that $\text{ex}(n, K_{3,3}) \geq \frac{1}{8}n^{4/3}$, provided n is sufficiently large. (Here, as before, you are free to take n to be as large as you want — for example, you can take $n \geq 100$.) **[7]**

(f) Write down a positive constant α such that for all $n \in \mathbb{N}$,

$$\text{ex}(n, K_{3,3}) \leq cn^{2-\alpha},$$

for some positive constant c. (You do not need to give the value of c.) **[3]**

End of Paper.