

Main Examination period 2018

MTH734U / MTHM012 / MTH712P: Topics in Probability and Stochastic Processes

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: Dr David Ellis

Question 1. [20 marks]

- (a) Let $\{N(t) : t \geq 0\}$ be a renewal process with interarrival times X_i (for $i \in \mathbb{N}$), which are independent, identically distributed random variables with mean $\mu = \mathbb{E}[X_1]$, where $0 < \mu < \infty$. Let $m(t) = \mathbb{E}[N(t)]$ for each $t \geq 0$. What does the Elementary Renewal Theorem say about $\lim_{t \rightarrow \infty} \frac{N(t)}{t}$? What does it say about $\lim_{t \rightarrow \infty} \frac{m(t)}{t}$? [3]
- (b) State the Renewal Reward theorem. [4]

Suppose that a smoke alarm holds one battery at a time. When a battery fails, it is replaced immediately. It is replaced with a brand-A battery with probability $2/5$ and with a brand-B battery with probability $3/5$. The lifetime of each brand-A battery (measured in years) has the exponential distribution with parameter $\frac{1}{3}$ years⁻¹, i.e. it has pdf

$$f_A(x) = \begin{cases} \frac{1}{3}e^{-x/3} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The lifetime of a brand-B battery (measured in years) has pdf

$$f_B(x) = \begin{cases} \frac{3}{4}x(2-x) & \text{if } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

The lifetimes of all bulbs are independent of one another.

- (c) Write down, or calculate, the expected lifetime of a brand-A battery. [1]
- (d) Find the expected lifetime of a brand-B battery. [2]
- (e) In the long run, what is the average number of battery-replacements per year? [3]
- (f) Suppose that a brand-A battery costs £3 and a brand-B battery costs £2. In the long run, what is the average cost per year of replacing batteries in the smoke alarm? [4]
- (g) Suppose now that you have a choice of either using *only* brand-A batteries, or *only* using brand-B batteries. Which strategy would be more cost-effective in the long run? Justify your answer. [3]

Question 2. [35 marks]

- (a) State the *Superposition Lemma* and the *Thinning Lemma* for Poisson processes. [5]
- (b) Show that if X and Y are independent Poisson random variables with means λ and μ respectively, then $X + Y$ has the Poisson distribution with mean $\lambda + \mu$. Explain how this fact can be used in the proof of the Superposition Lemma. [5]

Two archers, Alice and Bob, play a game where they both shoot arrows at a target. They both start shooting at the same time. Alice scores hits on the target according to a Poisson process of rate 2 per minute. Bob scores hits on the target according to a Poisson process of rate 1 per minute. Your answers to the following questions should be expressed in terms of powers of e , where necessary, but they should be simplified as much as possible in all other ways.

- (c) What is the probability that Alice has scored just one hit after 3 minutes? [2]
- (d) What is the probability that Alice has scored just one hit after 3 minutes and just four hits (in total) after 5 minutes? [3]
- (e) What is the probability that the total number of hits (by both Alice and Bob) is 2, after 3 minutes? [3]
- (f) Suppose that each of Alice's hits is a bull's-eye with probability $1/8$, independently of all other hits. What is the expected time until Alice scores her first bull's-eye? [3]
- (g) Suppose now that the game lasts for 2 minutes and that at the end of the game, the score is 4 hits to Alice and 2 hits to Bob. Conditional on this information,
- (i) What is the probability that Alice has scored just one hit after one minute of the game (i.e., halfway through the game)? [3]
- (ii) What is the probability that Bob is ahead of Alice after one minute of the game? [5]
- (iii) What is the expected time at which Alice scores her fourth hit? [6]

Question 3. [15 marks] A robot has just three speeds: 5 mph, 10 mph and 30 mph. When it changes speed to 30 mph, it remains at that speed on average for 2 hours before changing speed to either 10 mph (with probability $3/4$) or to 5 mph (with probability $1/4$). When it changes speed to 10 mph, it remains at that speed on average for 4 hours before changing speed to either 5 mph (with probability $1/2$) or to 30 mph (with probability $1/2$). When it changes speed to 5 mph, it remains at that speed on average for 1 hour before changing speed to either 10 mph (with probability $3/4$) or to 30 mph (with probability $1/4$).

- (a) What extra assumption do we need to make in order to model the changing speed of the robot as a semi-Markov process? [2]
- (b) Assume that the changing speed of the robot forms a semi-Markov process. Label the states as 1, 2 and 3 (in increasing order of speed). Write down the transition matrix P of the associated discrete-time Markov chain with the same state-space and transition-probabilities. [2]
- (c) Find an equilibrium distribution $\pi = (\pi_1, \pi_2, \pi_3)$ for the associated discrete-time Markov chain, and show that it is the unique equilibrium distribution, by solving the equations

$$\pi P = \pi, \quad \sum_{i=1}^3 \pi_i = 1,$$

- and showing that the solution is unique. [6]
- (d) Estimate the proportion of time the robot spends at each of the three speeds, over a very long time-period. [3]
- (e) Assume that the robot travels in a straight line without reversing direction. Estimate the average speed of the robot over a very long time-period. [2]

Question 4. [30 marks] Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain with generator matrix G , finite state space S , and time- t transition matrices $\{P(t) : t \geq 0\}$. Let $\pi = (\pi_i)_{i \in S}$ be a probability distribution on S .

- (a) Define (in terms of the generator matrix G , or otherwise) what it means for π to be an *equilibrium distribution* for the continuous-time Markov chain $\{X(t) : t \geq 0\}$. [2]
- (b) Define (in terms of the matrices $\{P(t) : t \geq 0\}$, or otherwise), what it means for π to be a *limiting distribution* for the Markov chain. [2]
- (c) Show that if π is an equilibrium distribution for the Markov chain, and $X(0)$ has distribution π , then $X(t)$ has distribution π for all $t > 0$. [3]

A small post office has two clerks. The customers in the post office who are waiting to be served, join a single queue. If there are at most two customers in the post office, the time until the arrival of the next customer is exponentially distributed with mean 5 minutes, but if there are three customers in the post office, no new customers enter. Each clerk can only serve one customer at a time, and the time taken to serve each customer is exponentially distributed with mean 10 minutes. After being served, a customer leaves.

- (d) Let $Y(t)$ denote the total number of customers in the post office, t minutes after it has opened. Find the generator matrix G for the continuous-time Markov chain $\{Y(t) : t \geq 0\}$, indicating how the rows and columns of G are indexed by the states. [6]
- (e) Find an equilibrium distribution for the continuous-time Markov chain in part (b), and show that it is the unique equilibrium distribution. [6]
- (f) Explain why any state of this Markov chain communicates with any other state. [2]
- (g) State, with justification, the limiting distribution of this Markov chain. (You may appeal to any standard theorem or fact from the course, without proving it.) [2]
- (h) Estimate the proportion of time for which there are three customers in the post office, in the long run. [2]
- (i) Suppose that when I enter the post office at 10 am, there is just one customer already there. What is the expected time at which both of us have finished being served? [5]

End of Paper.