Queen Mary
University of London

MTH734U / MTHM012 / MTH712P: Topics in Probability and Stochastic Processes

Duration: 3 hours

Date and time: 4th May 2016, 10:00–13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Dr Dudley Stark, Dr Jens Starke

 Turn Over

Question 1. Let $N(t)$, $t \geq 0$, be a continuous time renewal process with interoccurrence times $X_i > 0$ which are independent, identically distributed continuous random variables with common distribution $\mathbb{P}(X_i \leq x) = F(x)$. Let $S_0 = 0$ and let $S_n = X_1 + X_2 + \cdots + X_n$ be the waiting time until the occurrence of the nth event for $n \geq 1$. Suppose $\mu = \mathbb{E}(X_1) < \infty$. Let $M(t) = \mathbb{E}(N(t))$.

(a) State the definition of $N(t)$ in terms of the random variables S_n. [**3**]

(b) Prove that

$$M(t) = \sum_{n=1}^{\infty} F_n(t),$$

where $F_n(t) = \mathbb{P}(S_n \leq t)$. [**5**]

(c) The current life δ_t has limiting distribution

$$\lim_{t\to\infty} \mathbb{P}(\delta_t \leq x) = \frac{1}{\mu} \int_0^x [1 - F(u)]\, du.$$

Supposing that the X_i are Exponential(θ) distributed, find $\lim_{t\to\infty} \mathbb{P}(\delta_t \leq x)$. [**5**]

(d) Let $N_D(t)$ be the delayed renewal process for which X_1 has c.d.f. $G(x)$ and the X_i have c.d.f. $F(x)$ for all $i \geq 2$. Let $M_D(t) = \mathbb{E}(N_D(t))$ and let $g(x) = G'(x)$. Prove that

$$M_D(t) = \int_0^t g(x) M(t - x)\, dx + G(t).$$

[**7**]

Question 2.

(a) Given a semi-Markov process on states $\{1, 2, \dots, N\}$, suppose that when the process enters state i, it stays there a random amount of time having expectation μ_i after which it jumps to state j with probability $P_{i,j}$.

(i) State what is meant by the Markov chain associated to the semi-Markov process. **[2]**

(ii) State the equations for the stationary distribution of the Markov chain associated to the semi-Markov process. **[2]**

(iii) Suppose that $N = 4$ and $P_{1,2} = P_{2,3} = P_{3,2} = P_{4.4} = 1$. Find the general solution of the equations for the stationary distribution of the Markov chain determined by the $P_{i,j}$ and show that there is not a unique stationary distribution. **[5]**

(b) A truck driver regularly drives round trips from Birmingham to Manchester, from Manchester to Leeds, and then from Leeds back to Birmingham. The time it takes him to drive from Birmingham to Manchester is uniformly distributed between one and two hours. The time it takes him to drive from Manchester to Leeds is equally likely to be either one hour or 100 minutes. The time it takes him to drive from Leeds to Birmingham is equally likely to be either 80 minutes or two hours. The times his various trips take are all independent of each other. In the long run, what proportion of the time that his trips take is spent driving from Leeds to Birmingham? **[11]**

Question 3.

(a) (i) State what is meant by the memoryless property of the exponential distribution. **[2]**

(ii) Prove the memoryless property for the exponential distribution. **[4]**

(b) Customers arrive at a bank according to a Poisson process with rate 6 customers per hour.

(i) What is the probability that five customers arrive in the first hour after the bank opens? **[2]**

(ii) What is the probability that five customers arrive in the first hour after the bank opens and that three customers arrive in the second hour after the bank opens? **[4]**

(iii) Suppose it is known that five customers arrived in the first hour after the bank opened. What is the probability that exactly one of the five customers arrived during the first 20 minutes? Justify your answer using a theorem proved in lecture. **[8]**

Question 4.

(a) Let $\mathbf{G}$ be the generator of a continuous time Markov chain $X(t)$, with $t \geq 0$, and let $\mathbf{P}(t)$ be the matrix such that $\mathbf{P}(t)_{i,j} = \mathbb{P}(X(s+t) = j | X(s) = i)$.

(i) State the equation for $\mathbf{P}(t)$ in terms of $\mathbf{G}$. **[2]**

(ii) State the backwards and forwards Kolmogorov equations. **[2]**

(b) A small barbershop is operated by two barbers, each of which has a chair for a single customer. When two customers are in the store, its doors are locked so no more customers can enter. The remaining time, when there are fewer than two customers in the store, its doors are left open and potential customers arrive according to a Poisson process with rate of three per hour. Successive service times are independent exponential random variables with expectation of $\frac{1}{4}$ hour.

(i) Find the generator for $X(t)$, the number of customers in the shop. **[7]**

(ii) Find the limiting probabilies $\pi_i = \lim_{t\to\infty} \mathbb{P}(X(t) = i)$ for $i = 0, 1, 2$. **[7]**

(iii) In the long run, what is the average number of customers in the shop? **[2]**

Question 5. Let $B(t)$ be standard Brownian motion with $B(0) = 0$.

(a) State the distribution of $B(t)$ including any parameters. **[3]**

(b) Given a constant $t_0 > 0$, show that the process $\tilde{B}(t)$ defined by

$$\tilde{B}(t) = B(t + t_0) - B(t_0)$$

is identical to Brownian motion. **[8]**

(c) Let S_n be a symmetric random walk, defined by $S_n = \sum_{i=1}^{n} X_i$, where the X_i are independent and identically distributed random variables with distribution $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$.

(i) State what is meant by the Invariance Principle for $B(t)$. **[3]**

(ii) Given integers $\alpha, \beta > 0$, let $T(\alpha, \beta)$ be defined by

$$T(\alpha, \beta) = \min\{n \geq 0 : S_n = -\alpha \text{ or } S_n = \beta\}.$$

You may assume that $\mathbb{E}(T(\alpha, \beta)) = \alpha\beta$. Let $a, b > 0$ and let τ be defined by

$$\tau = \min\{t \geq 0 : B(t) = -a \text{ or } B(t) = b\}.$$

Use the Invariance Principle to determine $\mathbb{E}(\tau)$. **[6]**

End of Paper.