Queen Mary University of London

Main Examination period 2018

MTH716U/MTHM007: Measure Theory and Probability

Duration: 3 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Examiners: C. Joyner, I. Goldsheid

Throughout this exam the term **measurable** will be used to mean **Lebesgue measurable** and $\mathcal{M}$ will denote the collection of Lebesgue measurable subsets of $\mathbb{R}$. For all measurable sets $E \in \mathcal{M}$ we will denote $m(E)$ to be the corresponding Lebesgue measure of E.

Question 1. [25 marks]

(a) State the definition of a **null set**. **[3]**

(b) The Cantor set C is constructed by starting with $[0,1]$ and successively removing the middle third from each remaining interval, i.e. $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \ldots$ etc. and taking

$$C = \bigcap_{k=1}^{\infty} C_k.$$

Show that the Cantor set C is null. **[3]**

(c) State the definition of **outer measure** $m^*(A)$ of a set $A \subseteq \mathbb{R}$. **[3]**

(d) Prove that a set $A \subset \mathbb{R}$ is null **if and only if** its outer measure satisfies $m^*(A) = 0$. **[5]**

(e) Show that outer measure obeys **countable sub-additivity**, i.e.

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

[6]

(f) Show that for a set $A \subset \mathbb{R}$ and constants c, t, with $c \geq 0$, outer measure obeys

$$m^*(cA + t) = cm^*(A),$$

where $cA + t := \{cx + t : x \in A\}$. **[5]**

Question 2. [25 marks]

(a) State the definition of a **measurable set** $E \subseteq \mathbb{R}$. **[3]**

(b) Show that any **null set** is measurable. **[4]**

(c) The symmetric difference of two sets $A, B \subseteq \mathbb{R}$ is given by $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Show that if $A \in \mathcal{M}$ and $m(A \Delta B) = 0$ then $B \in \mathcal{M}$ and $m(A) = m(B)$.

You may use the monotonicity condition that for two sets $A, B \in \mathcal{M}$ such that $A \subseteq B$ the measure satisfies $m(A) \leq m(B)$. **[6]**

(d) Show that if $E_1, E_2 \subseteq \mathbb{R}$ are two **disjoint measurable sets** then the union $E_1 \cup E_2$ is also measurable and

$$m(E_1 \cup E_2) = m(E_1) + m(E_2).$$

[6]

(e) State the three properties for a collection $\mathcal{F}$ of subsets of Ω to be a σ-field? **[3]**

(f) Let us define the restriction of the collection of Lebesgue measurable sets $\mathcal{M}$ to a measurable set $B \in \mathcal{M}$ as

$$\mathcal{M}|_B := \{E \cap B : E \in \mathcal{M}\}.$$

Show that $\mathcal{M}|_B$ is a σ-field over B. **[3]**

Question 3. [20 marks]

(a) State the definition of a **measurable function** $f : \mathbb{R} \to \mathbb{R}$. **[3]**

(b) Show that if the set $f^{-1}((a,\infty)) = \{x : f(x) > a\}$ is measurable for all $a \in \mathbb{R}$ then the sets $f^{-1}([a,\infty)), f^{-1}((-\infty,a))$ and $f^{-1}((-\infty,a])$ are also measurable. **[5]**

(c) Let $E \subseteq \mathbb{R}$ be a measurable set and take the function

$$f(x) = \mathbf{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

Show that f is a measurable function. **[3]**

(d) Let $\mathcal{F}$ be a σ-field over Ω and $\mu : \mathcal{F} \to [0,\infty]$ a set function. What are the conditions needed for μ to be a measure? **[2]**

(e) What additional property is needed in Part (d) for μ to be a probability measure? **[1]**

(f) Let $\mathcal{F} = \mathcal{M}|_{[0,1]} = \{E \in \mathcal{M} : E \subseteq [0,1]\}$ be the collection of Lebesgue measurable sets $\mathcal{M}$ restricted to the interval $[0,1]$. Let $X : [0,1] \to \mathbb{R}$ be a random variable on the probability space $([0,1], \mathcal{F}, m)$. Find the σ-field generated by X when

(i) $X(\omega) = \mathbf{1}_{[0,\frac{1}{3})}(\omega) + \mathbf{1}_{[\frac{2}{3},1]}(\omega)$. **[3]**

(ii) $X(\omega) = \omega \mathbf{1}_{\mathbb{Q}}(\omega)$. **[3]**

Question 4. [30 marks]

(a) State the definition of a **simple function** ϕ and its **Lebesgue Integral** $\int_E \phi \, dm$ for a measurable set E. **[3]**

(b) For a non-negative simple function $\phi : \mathbb{R} \to \mathbb{R}$ and two disjoint measurable sets $E_1, E_2 \subseteq \mathbb{R}$ show that

$$\int_{E_1 \cup E_2} \phi \, dm = \int_{E_1} \phi \, dm + \int_{E_2} \phi \, dm.$$

[4]

(c) State the definition of the **Lebesgue Integral** $\int_E f \, dm$ for a **non-negative** measurable function f and measurable set E. **[2]**

(d) State the Monotone Convergence Theorem. **[4]**

(e) Let f be a non-negative measurable function and define the set function $\mu : \mathcal{M} \to [0, \infty]$ as

$$\mu(E) := \int_E f \, dm.$$

Using the Monotone Convergence Theorem show that μ is a measure on the measurable space $(\mathbb{R}, \mathcal{M})$. **[5]**

(f) Provide an example of a function f for which the measure μ in Part (e) is a probability measure on $\mathbb{R}$. **[2]**

(g) State the definition for a function f to be **Lebesgue integrable** over a measurable set E and define the corresponding Lebesgue integral $\int_E f \, dm$. **[3]**

(h) Give an example of a function f that is integrable over $E = [0, 1]$ but not over $E = [0, 5]$. **[3]**

(i) Show that for integrable functions f and g over $E \in \mathcal{M}$ the function $f + g$ is also integrable and that

$$\int_E (f + g) \, dm = \int_E f \, dm + \int_E g \, dm. \tag{1}$$

You may assume the equality (1) holds when f and g are **non-negative measurable** functions. **[4]**

End of Paper.