Queen Mary
University of London

M. Sci. Examination by course unit 2015

MTH716U: Measure Theory & Probability

Duration: 3 hours

Date and time: 14th May, 10:00-13:00

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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Examiner(s): N. Simm and C. Joyner

Throughout this exam the term *measurable* is used to mean *Lebesgue measurable*.

Question 1. (a) Given a subset $A \subseteq \mathbb{R}$, how is the *outer measure* $m^*(A)$ defined? [4]

(b) What does it mean to say that a subset $A \subseteq \mathbb{R}$ is *null*? [2]

(c) Briefly describe the construction of a set C which is both uncountable and null. Provide the proof that C is null. [4]

(d) Prove that if $A \subseteq B \subseteq \mathbb{R}$ then $m^*(A) \leq m^*(B)$. [4]

(e) If A and B are subsets of $\mathbb{R}$, prove that $m^*(A \cup B) \leq m^*(A) + m^*(B)$. [6]

(f) Making use of parts (d) and (e) above, or otherwise, prove that if the set $A \triangle B$ is null then $m^*(A) = m^*(B)$, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of the two sets A and B. [5]

Question 2. (a) State the definition of a *measurable* subset $E \subseteq \mathbb{R}$. [3]

(b) Prove that every null set is measurable (you may use without proof the results of parts (d) and (e) of question 1). [5]

(c) Let $E + t := \{x + t : x \in E\}$ where $t \in \mathbb{R}$. Prove that $E \subseteq \mathbb{R}$ is measurable if and only if $E + t$ is measurable. You may assume without proof that $m^*(A + t) = m^*(A)$ for any $A \subseteq \mathbb{R}$. [5]

(d) What does it mean to say that the collection $\mathscr{M}$ of all measurable subsets of $\mathbb{R}$ forms a σ-field? [3]

(e) What does it mean to say that Lebesgue measure $m : \mathscr{M} \to [0, \infty]$ is *countably additive*? [3]

(f) Let E_1 and E_2 be disjoint (*i.e.* $E_1 \cap E_2 = \emptyset$) and measurable subsets of $\mathbb{R}$. Prove that $E_1 \cup E_2$ is measurable and

$$m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2).$$

Again, you may use without proof the results of parts (d) and (e) of question 1. [6]

Question 3. (a) What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is *measurable*? [3]

(b) Show that if the set $f^{-1}((a,\infty)) = \{x : f(x) > a\}$ is measurable for any $a \in \mathbb{R}$ then so are the sets $f^{-1}((-\infty,a])$, $f^{-1}((-\infty,a))$ and $f^{-1}([a,\infty))$. [5]

(c) Using the condition that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable if $f^{-1}((a,\infty))$ is measurable for all $a \in \mathbb{R}$, show that the constant function, i.e. $f(x) = c$ for all $x \in \mathbb{R}$, is measurable. [4]

(d) Suppose the functions $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are all measurable. Show the combined function $F(x) = f(x) + g(x)\exp(h(x))$ is also measurable. You may use the fact that every open set in $\mathbb{R}^2$ decomposes into a countable union of rectangles. [6]

(e) Define the essential supremum and essential infimum of a measurable function $f : E \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty,\infty]$ denotes the extended real line. [3]

(f) Find the essential supremum of the following two functions

(i)

$$f(x) = \begin{cases} 1/x & x \in \mathbb{R}\backslash\{0\} \\ 0 & x = 0. \end{cases}$$

(ii)

$$g(x) = \begin{cases} \exp(x) & x \in \mathbb{Q} \\ \sin(x) & x \notin \mathbb{Q}. \end{cases}$$

[4]

Question 4. (a) Define what it means for a non-negative function $\varphi : \mathbb{R} \to \mathbb{R}$ to be simple and state the definition of its integral $\int_E \varphi dm$, where $E \subseteq \mathbb{R}$ is measurable. [2]

(b) For a non-negative simple function $\varphi : \mathbb{R} \to \mathbb{R}$ and two disjoint measurable sets $E_1, E_2 \subseteq \mathbb{R}$ show that [3]

$$\int_{E_1 \cup E_2} \varphi dm = \int_{E_1} \varphi dm + \int_{E_2} \varphi dm.$$

(c) Given a measurable set E state the definition of the integral $\int_E f dm$ for a non-negative measurable function $f : E \to \mathbb{R}$. [3]

(d) State Fatou's Lemma for a sequence of measurable functions $\{f_n\}$. [4]

(e) Give an example of a function for which Fatou's Lemma gives a *strict* inequality and explain why this is the case. [3]

(f) State the Monotone Convergence Theorem. [4]

(g) Suppose that $\{f_n\}$ and f are non-negative, measurable and $f_n \nearrow f$ almost everywhere. Using the Monotone Convergence Theorem show that for a measurable set E

$$\lim_{n\to\infty} \int_E f_n dm = \int_E f dm.$$

You may assume the statement in part (b) above also holds for measurable functions. [6]

End of Paper.