MTH714U / MTHM024: Group Theory

Duration: 3 hours

Date and time: 13 May 2016,   14.30h–17.30h

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks awarded are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): L. H. Soicher

Question 1.

(a) Let G be a group acting on a set Ω, and let $\alpha \in \Omega$. What is meant by the *orbit* $\mathrm{Orb}_G(\alpha)$ of α? What is meant by the *stabiliser* $\mathrm{Stab}_G(\alpha)$ of α? **[4]**

(b) State the Orbit-Stabiliser Theorem. **[4]**

(c) Let p be a prime, and let G be a group of order p^n for some positive integer n. Apply the Orbit-Stabiliser Theorem to prove that $Z(G) \neq \{1\}$ [where $Z(G) = \{g \in G : xg = gx \text{ for all } x \in G\}$]. **[5]**

(d) Let p be a prime. Apply the Fundamental Theorem of Abelian Groups to determine all the abelian groups of order p^3, up to isomorphism. [You do not need to justify your answer.] **[4]**

(e) Let p be a prime, and let G be a group of order p^3. Prove that either G is abelian or $|Z(G)| = p$. **[8]**

Question 2.

(a) What does it mean to say that a permutation of $\{1, \ldots, n\}$ is *even*? What is meant by the *symmetric group* S_n, what is meant by the *alternating group* A_n, and what is meant by a *normal subgroup* of a group? **[8]**

(b) Write down all the normal subgroups of the group S_4. [You do not need to justify your answer.] **[4]**

(c) Let G be a group, let H be a subgroup of G and let K be a normal subgroup of G. Apply the First Isomorphism Theorem to prove that

$$H/(H \cap K) \cong HK/K.$$

[You may assume, without proof, that $H \cap K$ is a normal subgroup of H and that HK is a subgroup of G containing K.] **[5]**

(d) Let $n \geq 5$. Determine, with proof, the normal subgroups of S_n. [You may assume, without proof, that if $n \geq 5$ then A_n is a simple group and $|A_n| = |S_n|/2$.] **[8]**

Question 3. Let G be a group.

(a) Define what is meant by an *automorphism* of G, and what is meant by an *inner automorphism* of G. **[4]**

(b) Prove that an inner automorphism of a group G really is an automorphism of G. **[6]**

(c) Apply Sylow's theorems on the existence and properties of Sylow p-subgroups to prove that the symmetric group S_5 has a transitive faithful action on a set of size 6. **[7]**

(d) Prove that the symmetric group S_6 has an automorphism which is not an inner automorphism. **[8]**

Question 4.

(a) Let G be a group. Define what is meant by the *commutator* $[g, h]$ of elements $g, h \in G$, and what is meant by the *commutator subgroup* (or *derived group*) G' of G. **[4]**

(b) State Iwasawa's Lemma. **[4]**

(c) Let F be a field, let $n > 1$, let $V = F^n$, let a be a non-zero vector in V and let $f : V \to F$ be a linear map with $af = 0$.

(i) Define what is meant by the *transvection* $T(a, f)$ on V, and what is meant by the *transvection group* $A(a)$. Define the group $\mathrm{SL}(n, F)$. **[6]**

(ii) Explain why $A(a)$ can be considered to be a subset of $\mathrm{SL}(n, F)$.
[You may assume, without proof, that a transvection from V to V is a linear map. You are not required to prove that $A(a)$ is a subgroup of $\mathrm{SL}(n, F)$.] **[4]**

(iii) Let $w = (0, 1) \in F^2$. Prove that if $|F| > 3$ then $A(w)$ is contained in the commutator subgroup $\mathrm{SL}(2, F)'$ of $\mathrm{SL}(2, F)$. **[7]**

End of Paper.