Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH6142: Complex Networks

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

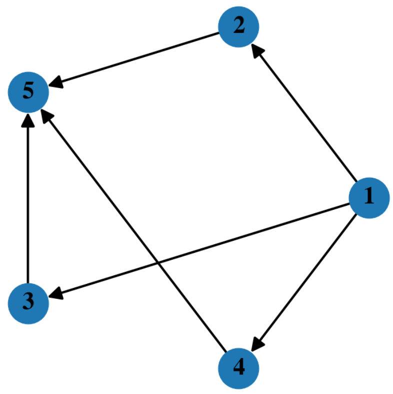
When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: V. Latora, G. Bianconi

Question 1 [40 marks].
Consider the directed network G, with $N = 5$ nodes and $L = 6$ links, shown in figure.



a) Write the adjacency matrix $\mathbf{A}$ of the network (*Use the convention that entry a_{ij} is equal to 1 if node j points to node i*). Is matrix $\mathbf{A}$ symmetric? **[4]**

b) Determine the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}, k_5^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}, k_5^{out}\}$ of the network. Find the nodes with the largest in-degree and the ones with the largest out-degree. **[6]**

c) Determine the in-degree distribution $P^{in}(k)$ and the out-degree distribution $P^{out}(k)$. **[4]**

d) State the definition of the Katz centrality x_i of a node i of a network. Find the Katz-centrality for all the nodes of the network G. **[7]**

e) How many weakly and how many strongly connected components are there in the network G? Which are the nodes belonging to each one of these components? **[3]**

f) Consider now the underlying undirected network G^u of G, i.e. the network obtained from G by neglecting the directions of the links. Write the adjacency matrix $\mathbf{A}^u$ of G^u. How many links has network G^u? **[4]**

g) Write the $N \times N$ distance matrix $\mathbf{d}$ of network G^u (*Remember that entry d_{ij} indicates the length of the shortest paths between node i and j*). What it the diameter D of G^u? **[4]**

h) Determine the degree distribution $P(k)$ of network G^u. **[2]**

i) State the definition of the betweenness centrality b_i of a node i of an undirected network. Evaluate the betweenness centrality b_1 of node 1 and the betweenness centrality b_4 of node 4 in network G^u. **[6]**

Question 2 [35 marks].
Let $n_0 = 10$, $m = 3$ and $a > 0$, and consider the following model to grow simple networks.
At time $t = 1$ we start with a complete network with n_0 nodes.
At each time step $t > 1$ a new node is added to the network. The node arrives together with m new links, which are connected to m different nodes already present in the network. The probability Π_i that a new link is connected to node i is:

$$\Pi_i = \frac{k_i + a}{Z} \qquad \text{with} \;\; Z = \sum_{j=1}^{N(t-1)} (k_j + a)$$

where k_i is the degree of node i, and $N(t-1)$ is the number of nodes in the network at time $t-1$.

(a) Find an expression for the number of nodes, $N(t)$, and the number of links, $L(t)$, in the network as a function of time t. [**4**]

(b) What is the average node degree $\langle k \rangle$ at time t? What is the average node degree in the limit $t \to \infty$? [**4**]

(c) Find an expression for the value of Z as a function of time t. [**3**]

(d) Write down the differential equation governing the time evolution of the degree k_i of node i for $t \gg 1$ in the mean-field approximation. Solve this equation with the initial condition $k_i(t_i) = m$, where t_i is the time of arrival of node i. [**7**]

(e) Derive the degree distribution $P(k)$ of the network for $t \gg 1$ in the mean-field approximation. [**7**]

(f) Does the model produce scale-free networks? If so, what is the value of the degree exponent γ? [**4**]

(g) Write down the master equation of the model, i.e. the equation that describes the evolution of the average number $N_k(t)$ of nodes that at time t have degree k. [**6**]

End of Paper.