Queen Mary
University of London

Main Examination period 2019

MTH6142: Complex Networks

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any unauthorised notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it will be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: G. Bianconi, V. Nicosia

Question 1. [40 marks]
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 5$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

a) Draw the network. Is the network directed or undirected? (*Explain your answer.*) **[7]**

b) How many weakly and how many strongly connected components are there in the network? Which are the nodes belonging to each one of these components? **[4]**

c) Is there an in-component? If yes, which are the nodes belonging to it? **[3]**

d) Is there an out-component? If yes, which are the nodes belonging to it? **[3]**

e) Determine the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}, k_5^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}, k_5^{out}\}$. **[4]**

f) Determine the in-degree distribution $P^{in}(k)$ and the out-degree distribution $P^{out}(k)$. **[4]**

g) Calculate the $N \times N$ matrix $\mathbf{d}$ of elements $d_{ij} \in \mathbb{N}_0 \cup \{\infty\}$ indicating the shortest distance of node j from node i. **[5]**

h) Calculate the eigenvector centrality x_i of each node $i = 1, 2, \ldots, N$ of the network with adjacency matrix $\mathbf{A}$ defined above.
To this end start from the initial guess $\mathbf{x}^{(0)} = \frac{1}{N}\mathbf{1}$ where $\mathbf{1}$ is the N-dimensional column vector of elements $1_i = 1\ \forall i = 1, 2 \ldots, N$. Consider the iteration

$$\mathbf{x}^{(n)} = \mathbf{A}\mathbf{x}^{(n-1)},$$

for $n \in \mathbb{N}$.
Finally, calculate the eigenvector centrality x_i of each node i of the network by finding the limit

$$x_i = \lim_{n\to\infty} \frac{x_i^{(n)}}{\sum_{j=1}^{N} x_j^{(n)}}.$$

[10]

Question 2. [25 marks]
Giant component of the random graph.
Consider a random graph ensemble $\mathbb{G}(N, p)$ formed by all networks of N nodes with each pair of nodes connected with probability p.
Take

$$p = \frac{c}{N-1}$$

with $c > 0$ indicating the average degree of the network.
Let S indicate the probability that a node is in the giant component.
A node i is not in the giant component of a random graph if for every other node j of the graph either one of the following events occurs:
i) i is not linked to j;
ii) i is linked to j but j doesn't belong to the giant component.

a) Show that in the large network limit $N \gg 1$, the probability S satisfies the equation

$$S = 1 - e^{-cS},$$

where c is assumed to be independent of the network size N. **[7]**

b) Show that the critical average degree for having a giant component in the limit of large N is $c = 1$. **[10]**

c) Show that the average degree c that ensures that the random network is connected, i.e. it is formed by a single connected component, is approximately given by

$$c \simeq \ln(N).$$ **[8]**

Question 3. [35 marks]
Growing network model
Consider the following growing network model with preferential attachment in which each node i is assigned an *attractiveness* $a = 3$.
Let $N(t)$ denote the total number of nodes at time t.
At time $t = 1$ the network is formed by two nodes joined by a link.

- At every time step a new node joins the network. Every new node has initially two links that connect it to the rest of the network.
- At every time step $t > 1$ each new link of the new node is attached to an existing node i of the network chosen with probability Π_i given by

$$\Pi_i = \frac{(k_i + a)}{Z},$$

where

$$Z = \sum_{j=1,\ldots,N(t-1)} (k_j + a).$$

a) Calculate the total number of nodes $N(t)$ and the total number of links $L(t)$ at time t. **[2]**

b) Calculate the value of the average degree $\langle k \rangle$ at any given time t and in the limit $t \to \infty$. **[3]**

c) Derive the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i in the mean-field approximation and in the limit $t \gg 1$. **[7]**

d) Derive the degree distribution $P(k)$ of the network for large times, i.e. $t \gg 1$, in the mean-field approximation. **[7]**

e) Is this network scale-free? (*Explain your answer*). **[3]**

f) Write the master equation for the average number $N_k(t)$ of nodes that at time t have degree k. **[2]**

g) Solve the master equation obtained in part f) and derive the corresponding degree distribution $P(k)$. **[11]**

End of Paper.