Main Examination period 2018

MTH6142 / MTH6142P: Complex networks

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Question 1. [40 marks]

Structural properties of a given network.
Consider the adjacency matrix $\mathbf{A}$ of a network of size $N = 5$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Draw the network. Is the network directed or undirected? (*Explain your answer.*) **[7]**

b) How many weakly and how many strongly connected components are there in the network? Which are the nodes belonging to each one of these components? **[4]**

c) Is there an in-component? If yes, which are the nodes belonging to it? **[3]**

d) Is there an out-component? If yes, which are the nodes belonging to it? **[3]**

e) Determine the in-degree sequence $\{k_1^{in}, k_2^{in}, k_3^{in}, k_4^{in}, k_5^{in}\}$ and the out-degree sequence $\{k_1^{out}, k_2^{out}, k_3^{out}, k_4^{out}, k_5^{out}\}$. **[4]**

f) Determine the in-degree distribution $P^{in}(k)$ and the out-degree distribution $P^{out}(k)$. **[4]**

g) Calculate the $N \times N$ matrix $\mathbf{d}$ of elements $d_{ij} \in \mathbb{N}_0 \cup \{\infty\}$ indicating the shortest distance of node i from node j. **[5]**

h) Calculate the eigenvector centrality x_i of each node $i = 1, 2, \ldots, N$ of the network with adjacency matrix $\mathbf{A}$ defined above.
To this end start from the initial guess $\mathbf{x}^{(0)} = \frac{1}{N}\mathbf{1}$ where $\mathbf{1}$ is the N-dimensional column vector of elements $1_i = 1\ \forall i = 1, 2 \ldots, N$. Consider the iteration

$$\mathbf{x}^{(n)} = \mathbf{A}\mathbf{x}^{(n-1)},$$

for $n \in \mathbb{N}$.
Finally, calculate the eigenvector centrality x_i of each node i of the network by finding the limit

$$x_i = \lim_{n\to\infty} \frac{x_i^{(n)}}{\sum_{j=1}^{N} x_j^{(n)}}.$$

[10]

Question 2. [25 marks]

Uncorrelated networks
Consider an uncorrelated network with degree distribution $P(k)$.

a) Express in terms of the degree distribution $P(k)$, the probability $q(k)$ that, by following a link, we reach a node of degree k.
Show that the average degree k_{nn} of the neighbours of a node is given by

$$k_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

where $\langle \ldots \rangle$ denotes the average over the degree distribution $P(k)$. **[3]**

b) Under which conditions it is true that

$$k_{nn} > \langle k \rangle \, ,$$

i.e. the average degree of the neighbours of a node is larger that the average degree of the network? **[4]**

c) Under which conditions it is true that

$$k_{nn} = \langle k \rangle \, ,$$

i.e. the average degree of the neighbours of a node is equal to the average degree of the network? **[1]**

d) Under which conditions it is true that

$$k_{nn} < \langle k \rangle \, ,$$

i.e. the average degree of the neighbours of a node is less than the average degree of the network? **[1]**

e) Consider a random Poisson network with average degree $\langle k \rangle = 3$. Calculate k_{nn} and verify that $k_{nn} > \langle k \rangle$. **[5]**

f) Consider an infinite power-law network with degree distribution $P(k) = Ck^{-\gamma}$ with $\gamma = 2.2$ and $k \geq 1$.
Calculate $\langle k \rangle$, $\langle k^2 \rangle$ and k_{nn} in the continuous approximation. **[10]**

g) For the power-law network defined in part f), is $k_{nn} > \langle k \rangle$? **[1]**

Question 3. [35 marks]

A growing network model with attractiveness of the nodes
Consider the following growing network model in which each node i is assigned an *attractiveness* $a_i \in \mathbb{N}^+$ drawn from a distribution $\pi(a)$.
Let $N(t)$ denote the total number of nodes at time t.
At time $t = 0$ the network is formed by two nodes joined by a link.

- At every time step a new node joins the network. Every new node has initially a single link that connects it to the rest of the network.
- At every time step t the link of the new node is attached to an existing node i of the network chosen with probability Π_i given by

$$\Pi_i = \frac{a_i}{Z},$$

where

$$Z = \sum_{j=1,\ldots,N(t-1)} a_j.$$

a) Calculate the total number of nodes $N(t)$ and the total number of links $L(t)$ at time t. **[4]**

b) What is the average degree $\langle k \rangle$ of the network at time t? **[2]**

c) Assume that

$$Z \simeq \bar{a}t,$$

where $\bar{a}$ indicates the average of a over the distribution $\pi(a)$.
Derive the time evolution $k_i = k_i(t)$ of the average degree k_i of a node i in the mean-field approximation. **[9]**

d) Assume that

$$\pi(a) = \begin{cases} 1 & \text{for} \quad a = 1, \\ 0 & \text{for} \quad a \neq 1, \end{cases}$$

and that $Z \simeq \bar{a}t$.
Derive the degree distribution $P(k)$ of the network for large times, i.e. $t \gg 1$, in the mean-field approximation. **[7]**

e) Under the same hypothesis as in part d) write the master equation for the average number $N_k(t)$ of nodes that at time t have degree k. **[3]**

f) Solve the master equation obtained in part e) and derive the corresponding degree distribution $P(k)$. **[10]**

End of Paper.