Queen Mary
University of London

Main Examination period 2022 – January – Semester A

MTH6141: Random Processes

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

IFoA exemptions. For actuarial students, this module counts towards IFoA actuarial exemptions. To be eligible for IFoA exemption, **your must submit your exam within the first 3 hours of the assessment period**.

Examiners: Robert Johnson, Mark Jerrum

In this exam $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Question 1 [25 marks]. Let $(X_0, X_1, X_2, \ldots,)$ be the discrete-time, homogeneous Markov chain on state space $S = \{1, 2, 3, 4\}$ with $X_0 = 1$ and transition matrix

$$\begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 0 & 0 & 3/4 & 1/4 \\ 1/4 & 3/4 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \end{pmatrix}$$

(a) Find the following probabilities:

(i) $\mathbb{P}(X_7 = 2 \mid X_6 = 1)$ **[3]**

(ii) $\mathbb{P}(X_2 \neq 1 \mid X_0 = 1)$ **[3]**

(b) Explain why the chain has a limiting distribution. Your explanation should refer to any results from lectures that you use. **[3]**

(c) Explain how to find this limiting distribution. Your explanation should refer to any results from lectures that you use. **[3]**

(d) What is the limiting distribution of this Markov chain? **[5]**

(e) For each of the following, say whether the limiting distribution enables you to write down a good approximation to it valid for large t. If the answer is Yes give the numerical value of this approximation; if the answer is No indicate briefly how you would calculate it by a different method (you do not need to give all the details or the numerical value):

(i) The probability that the chain is in an odd-numbered state at time t. **[2]**

(ii) The expectation of the proportion of time spent in state 4 up to time t. **[2]**

(iii) The expectation of the time of the third visit (including the one at time 0) to state 1. **[2]**

(iv) The probability that the chain visits state 4 before state 3. **[2]**

Question 2 [25 marks].

(a) Give an example of a statement involving recurrence and transience which holds for chains with finite state space but not for chains with infinite state space. [**4**]

Let $(X_0, X_1, X_2, \ldots)$ be the discrete-time, homogeneous Markov chain on state space $S = \mathbb{Z}$ with $X_0 = 1$ and transition probabilities

$$\begin{aligned} p_{i,i+2} = p_{i,i-2} &= 1/2 \text{ if } i \geqslant 4 \text{ is even} \\ p_{2,4} &= 1 \\ p_{0,0} &= 1 \\ p_{i,i+2} = p_{i,i-2} = p_{i,i-1} &= 1/3 \text{ if } i > 0 \text{ is odd} \\ p_{i,i-1} = p_{i,-1} &= 1/2 \text{ if } i \leqslant -1 \end{aligned}$$

(b) Sketch a representative portion of the transition graph for this chain. [**6**]

(c) Give a general description of how the chain will evolve. [**4**]

(d) Write down the communicating classes of this chain. [**5**]

(e) Classify each state as being transient, null recurrent or positive recurrent giving brief reasons. [**6**]

(You may use properties of the Markov chains we discussed as examples in lectures without proof provided you state clearly what you are refering to.)

Question 3 [25 marks]. In this question you may use properties of the Poisson process as given in lectures provided that you state which properties you are using.
I am counting bicycles passing my house. Each one is either a solo bike with a single rider, or a tandem with two riders. Solo bikes passing my house form a Poisson process of rate 5 per minute; tandems passing my house form a Poisson process of rate 1 per minute. These two Poisson processes are independent.

(a) What can you say about the process which counts the total number of bicycles passing my house? Justify your answer. [**3**]

(b) What is the expectation of the time that I wait until the third bicycle passes? [**3**]

(c) Calculate the probablility that:

(i) Exactly two tandems pass my house in the first t minutes. [**3**]

(ii) Exactly two bicycles pass my house in the first t minutes. [**3**]

(iii) Exactly two riders pass my house in the first t minutes. [**3**]

(d) Is the process which counts the total number of riders passing my house a Possion process? Justify your answer. [**2**]

(e) Is the process which counts the total number of riders passing my house a birth process? Justify your answer. **[2]**

(f) Let $R(t)$ be the number of riders who pass my house in the first t minutes and define

$$p_{i,j}(t) = \mathbb{P}(R(t+s) = j \mid R(s) = i)$$

Use the Chapman-Kolmogorov relations to derive the forward differential equation for $p'_{0,j}(t)$ valid for all $j \geq 2$. **[6]**

Question 4 [25 marks].

(a) A football team plays a sequence of matches each of which they either win, lose, or draw. At the start of the season they have 0 points. For each match, they score 3 points if they win, 1 point if they draw and 0 points if they lose. Suppose that each match is won with probablilty p_w and lost with probability p_l, independently of all other matches.

(i) You want to model the number of points accumulated as the season progresses as a discrete-time, homogeneous Markov chain $(X_0, X_1, \ldots)$. Describe how you would do this. Your description should specify what the X_i are in words, the state space, and the transition probabilities. **[6]**

(ii) Suppose that we want to allow different probabilities of winning for different matches (rather than a single p_w). Will your model still be a discrete-time, homogeneous Markov chain. Explain briefly what (if anything) changes. **[4]**

(iii) Suppose that we want to allow the probability of winning a match to depend on whether the previous match was won or lost. Specifically, if the previous match was won there will be a higher probability of winning the current match than if the previous match was lost. Will your model still be a discrete-time, homogeneous Markov chain. Explain briefly what (if anything) changes. **[4]**

(b) A long steel cable contains flaws which occur randomly along its length at a constant rate of α per metre.

(i) Give a reasonable way of modelling the location of the flaws as a stochastic process. **[4]**

(ii) Say in words what the interarrival times for this process represent. **[3]**

(iii) Each flaw is detected by an inspector with probability β independently of all other flaws. What can you say about the process describing the location of the undetected flaws? What concept does this illustrate? **[4]**

End of Paper.