Queen Mary
University of London

Main Examination period 2021 – January – Semester A

MTH6141: Random Processes

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **3 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

You are expected to spend about **2 hours** to complete the assessment, plus the time taken to scan and upload your work. Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

IFoA exemptions. For actuarial students, this module counts towards IFoA actuarial exemptions. For your submission to be eligible for IFoA exemptions, **you must submit within the first 3 hours of the assessment period**.

Examiners: S. Sodin, M. Jerrum

Please justify your work. Numerical answers should be presented in exact form (without approximation).

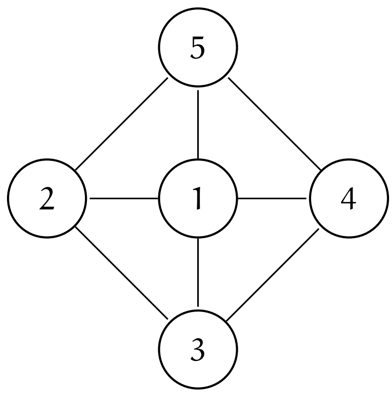
Question 1 [25 marks]. There are five light bulbs in a room, and no other sources of light. Each light bulb can either be on or off. Every minute, one of the five light bulbs is chosen at random (each is chosen with probability $1/5$), and its switch is flipped (if it was on, it is turned off, and if it was off, it is turned on).

(a) Show how to model the level of light in the room (after n minutes) as a Markov chain with six states. Draw the transition graph and write down the transition matrix. **[10]**

(b) Show that this Markov chain does not have a limiting distribution. **[5]**

(c) In the long run, which proportion of time are exactly 3 bulbs on? **[10]**

Question 2 [25 marks]. Consider the following combinatorial graph:



(a) Write down the transition matrix of the corresponding Markov chain. **[5]**

(b) Suppose the Markov chain starts from the state 2. What is the expected time until it reaches one of the vertices $1, 5$? **[10]**

(c) Suppose the Markov chain starts from the state 2. What is the probability that the state 5 is first visited before the state 1? **[10]**

Question 3 [25 marks]. Sasha receives emails from MTH6141 students according to a Poisson process of rate 3/hour. In addition, Sasha receives emails from his colleagues according to a Poisson process of rate 2/hour.

(a) Sasha enters his office and reads all the new emails in his mailbox. How long on average will he have to wait until the next one arrives? **[10]**

(b) What is the probability that from 9am to 11am, he receives four emails from students and two – from colleagues? **[10]**

(c) What is the conditional probability that he will receive 5 emails from students from 3pm to 4pm, given that two such emails arrive between 3:30pm and 4pm? **[5]**

Please justify your answers!

Question 4 [25 marks]. A gnome hunter needs to find two gnomes hidden in the room. Assume that the probability to find a gnome in a short time interval of length h is equal to $2h + o(h)$. Independently of that, a gnome found by the hunter can escape and hide at a new location in the room; the probability that this occurs in a time interval of length h is equal to $h + o(h)$. However, once the hunter holds both gnomes, he ties them so that they can no longer escape.

(a) Construct a continuous-time Markov chain describing this process. Write down the infinitesimal generator and the initial distribution. **[10]**

(b) What is the distribution of the time until the hunter manages to find the first gnome? **[5]**

(c) Let $p_n(t)$ be the probability that the hunter is holding n gnomes at time t $(n = 0, 1, 2)$. Write down a system of differential equations satisfied by these three functions. **[5]**

(d) What is the probability that the hunter will find both gnomes by time t? **[5]**

End of Paper.