Main Examination period 2017

MTH6141 / MTH6141P: Random Processes

Duration: 2 hours

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

You should attempt ALL questions. Marks available are shown next to the questions.

Calculators are not permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.

Complete all rough work in the answer book and cross through any work that is not to be assessed.

Possession of unauthorised material at any time when under examination conditions is an assessment offence and can lead to expulsion from QMUL. Check now to ensure you do not have any notes, mobile phones, smartwatches or unauthorised electronic devices on your person. If you do, raise your hand and give them to an invigilator immediately.

It is also an offence to have any writing of any kind on your person, including on your body. If you are found to have hidden unauthorised material elsewhere, including toilets and cloakrooms, it shall be treated as being found in your possession. Unauthorised material found on your mobile phone or other electronic device will be considered the same as being in possession of paper notes. A mobile phone that causes a disruption in the exam is also an assessment offence.

Exam papers must not be removed from the examination room.

Examiners: I. Goldsheid

Question 1. [22 marks] Let $X = (X_0, X_1, X_2, \ldots)$ be a Markov chain with state space S, transition probabilities $\{p_{i,j} : i, j \in S\}$, and initial distribution $\boldsymbol{\mu}^{(0)} = (\mu_i^{(0)} : i \in S)$. (You are reminded that $\mu_i^{(0)} = \mathbb{P}(X_0 = i)$.)

(a) State the definition of a **Markov chain**. **[3]**

(b) Prove that

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_t = i_t) = \mu_{i_0}^{(0)} p_{i_0,i_1} p_{i_1,i_2} \ldots p_{i_{t-2},i_{t-1}} p_{i_{t-1},i_t}.$$

[6]

(c) Define what it means to say that X is **irreducible**. Define what it means to say that X is **regular**. **[4]**

(d) What does it mean to say that a probability vector $\mathbf{w} = (w_i : i \in S)$ is a **limiting distribution** of X? **[3]**

(e) Prove that if $S = \{1, 2, ..., n\}$ and $\mathbf{w} = (w_1, w_2, ..., w_n)$ is a limiting distribution of X then $\mathbf{w}$ is the unique equilibrium distribution of X.

Hint: recall that for any initial distribution $\boldsymbol{\mu}^{(0)}$, we have $\boldsymbol{\mu}^{(t)} = \boldsymbol{\mu}^{(0)} P^t \to \mathbf{w}$ as $t \to \infty$. You can use this fact without proof. **[6]**

Question 2. [26 marks] Consider a Markov chain $Y = (Y_0, Y_1, Y_2, \ldots)$ with state space $\{1, 2, 3, 4\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

(a) Is Y irreducible? Justify your answer. **[3]**

(b) Is Y regular? Justify your answer. **[3]**

(c) Find the equilibrium distribution of Y. **[9]**

(d) Show that Y has no limiting distribution. **[6]**

(e) State the general theorem which allows one to compute, in terms of the equilibrium distribution, the proportion of time spent by a Markov chain in state $i \in S$ in the long run. **[4]**

(f) Use this theorem to compute the proportion of time spent by Y in state 2 in the long run. **[1]**

Question 3. [19 marks] Let $(X_0, X_1, X_2, \ldots)$ be a Markov chain with state space S and transition probabilities $\{p_{i,j} : i, j \in S\}$.

(a) Define what it means for a state $i \in S$ to be **transient**, in terms of the return probability $f_{i,i}$. [**2**]

(b) State a condition, in terms of the t-step transition probabilities $p_{i,i}^{(t)}$ alone, which is equivalent to the state i being:

(i) Transient. [**2**]

(ii) Recurrent. [**1**]

(iii) Null recurrent. [**2**]

(iv) Positive recurrent. [**2**]

(c) Prove that if states $i,\ j \in S$ intercommunicate then i is transient if and only if j is transient. [**10**]

Question 4. [13 marks]

(a) State the Superposition Lemma for Poisson processes. [**4**]

Consider now the following situation. Buses arrive at a bus stop according to a Poisson process $(X(t),\ t \geqslant 0)$ of rate 9 per hour. You arrive at the bus stop and take the first bus that arrives.

(b) What is the probability that your waiting time is less than 6 minutes? [**4**]

(c) Suppose that the first bus is full, so you have to take the second bus instead. Find the probability that your total waiting time is less than 12 minutes. [**5**]

Question 5. [20 marks]

(a) Let $(X(t) : t \geq 0)$ be a continuous-time birth process with state space $\mathbb{N} \cup \{0\}$ and with birth parameters $(\lambda_0, \lambda_1, \lambda_2, \ldots)$. State the theorem describing the necessary and sufficient conditions for explosion of the birth process. **[5]**

Consider now a birth process $X(t) : t \geq 0$ describing the number of individuals in a population at time t (measured in seconds). Individuals in this population reproduce according to the following rule. At time 0, there are two individuals. At any time t, if there are m individuals in the population, then each of the $\binom{m}{2}$ possible pairs give birth to offspring according to a Poisson process of rate 1 per second. (All Poisson processes are independent of one another.)

(b) Write down the birth parameters of this process. Justify your answer. **[5]**

(c) Find the expectation of the time of the third birth. **[6]**

(d) Does this process explode? Justify your answer. **[4]**

End of Paper.