Queen Mary
University of London

# B. Sc. Examination by course unit 2015

## MTH6141: Random Processes

**Duration: 2 hours**

**Date and time: 15th May 2015, 14:30–16:30**

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**Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.**

**You should attempt ALL questions. Marks awarded are shown next to the questions.**

**Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offence.**

Complete all rough workings in the answer book and **cross through any work that is not to be assessed**.

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**Exam papers must not be removed from the examination room.**

**Examiner(s): David Ellis**

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**Question 1.** Parts (a)-(e) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where as usual, the $i$th row (and the $i$th column) of $P$ corresponds to state $i$, for each $i \in \{1, 2, 3, 4, 5\}$.

(a) Is the Markov chain $(X_0, X_1, X_2, \ldots)$ irreducible? Justify your answer. [**3**]

(b) Is it regular? Justify your answer. [**3**]

(c) Find an invariant distribution for the Markov chain. [**4**]

(d) State whether or not the invariant distribution you found in part (d) is the *unique* invariant distribution. [**1**]

(e) Does the Markov chain have a limiting distribution? Justify your answer. [**3**]

Parts (f)-(i) of this question are about a Markov chain $(Y_0, Y_1, Y_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(f) Give an argument to show that the Markov chain $(Y_0, Y_1, Y_2, \ldots)$ is regular. [**3**]

(g) Find an invariant distribution for the Markov chain. [**5**]

(h) State whether or not the invariant distribution you found in part (h) is the *unique* invariant distribution. [**1**]

(i) Does the Markov chain have a limiting distribution? If so, what is it? [**2**]

**Question 2.** (a) Define what it means for a state to be an *absorbing state* of a Markov chain. [**2**]

Parts (b)-(d) of this question are about a Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$, and transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) List the absorbing states of the Markov chain $(X_0, X_1, X_2, \ldots)$. [**1**]

(c) Using the method of 'conditioning on the first step', or otherwise, find the probability that the Markov chain is eventually absorbed in state 1, given that $X_0 = 2$. Show your working. [**6**]

(d) Using part (c), or otherwise, write down the probability that the Markov chain is eventually absorbed in state 6, given that $X_0 = 2$. [**2**]

Parts (e)-(h) of this question are about a game. I play this game as follows. I start with a bucket containing eight balls: four green balls, three red balls and one yellow ball. Every minute, I put my hand into the bucket and I pick out one of the remaining balls at random (each ball is chosen with the same probability). If it is green, I put it back in the bucket, but if it is red or yellow, I keep it. The game ends either when I have picked out all three of the red balls, or when I have picked out the yellow ball. If I pick out all three of the red balls before I have picked out the yellow ball, then I win the game, but if I pick out the yellow ball before I have picked out all three of the red balls, then I lose the game.

(e) This game can be modelled as a Markov chain with five states (where two of the states are absorbing states). Write down what these five states are. (In other words, write down which physical situations each of the five states corresponds to.) [**3**]

(f) Draw a transition graph for this Markov chain. [**3**]

(g) Write down a transition matrix for this Markov chain. (Indicate clearly which rows of the matrix correspond to which states.) [**2**]

(h) By the method of conditioning on the first step, or otherwise, find the expected length of time until the game ends. Show your working. [**6**]

**Question 3.** Parts (a)-(d) of this question are about a general Markov chain $(X_0, X_1, X_2, \ldots)$ with state space $S$.

(a) Define what it means for a state $i \in S$ to be *recurrent*, in terms of the return probability $f_i$. [**2**]

(b) State a condition, in terms of the $t$-step transition probabilities $p_{i,i}^{(t)}$ alone, which is equivalent to the state $i$ being recurrent. [**2**]

(c) Define what it means for two states $i, j \in S$ to *intercommunicate* $(i \leftrightarrow j)$, in terms of probabilities. [**2**]

(d) Using part (b), or otherwise, show that if two states $i, j \in S$ intercommunicate, then $i$ is recurrent if and only if $j$ is recurrent. [**5**]

Parts (e)-(i) of this question are about a Markov chain $(Y_0, Y_1, Y_2, \ldots)$ with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(e) Draw a transition graph for this Markov chain. [**2**]

(f) Give a formula for $f_3^{(t)}$ (the probability that the first return to the state 3 is at time $t$, given that we start in state 3), in terms of $t$. Hence, or otherwise, show that the state 3 is recurrent. [**4**]

(g) Show that the state 1 is transient. [**3**]

(h) List the communicating classes of the Markov chain. [**3**]

(i) For each of the five states of the Markov chain, write down whether that state is recurrent or transient. [**2**]

**Question 4.** (a) Let $(X(t): t \geq 0)$ be a continuous-time stochastic process, and let $\lambda > 0$. Define what it means for $(X(t): t \geq 0)$ to be a *Poisson process with rate* $\lambda$. [**5**]

Parts (b)-(g) of this question are about an experiment with a particle detector, which is switched on at time 0, and then left on forever. Alpha particles arrive at the detector according to a Poisson process with rate 2 per hour. Beta particles arrive at the detector according to an independent Poisson process with rate 1 per hour. Your answers to parts (b)-(g) should be expressed in terms of powers of $e$ (where necessary), but they should be simplified in all other ways.

(b) What is the probability that exactly 3 alpha particles arrive at the detector in the first hour of the experiment? [**3**]

(c) What is the probability that the total number of particles (alpha particles plus beta particles) which arrive in the first hour, is 3? [**4**]

(d) Given that exactly 3 alpha particles arrive during the first hour, what is the probability that at least one alpha particle arrives during the second hour? [**3**]

(e) Given that exactly 3 alpha particles arrive during the first hour, what is the probability that exactly 2 alpha particles arrived during the first 30 minutes? [**3**]

(f) Given that exactly 3 alpha particles and exactly 2 beta particles arrive during the first hour, what is the probability that more beta particles than alpha particles arrived during the first 30 minutes? [**4**]

(g) Given that exactly one alpha particle arrived in the first thirty minutes, what is the probability that it actually arrived in the first five minutes? [**3**]

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**End of Paper.**