Main Examination period 2020 – January – Semester A

MTH6140 / MTH6140P: Linear Algebra II

Duration: 2 hours

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You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Examiners: T. W. Müller, M. Jerrum

Question 1 [23 marks].
In this question, V is a finite-dimensional vector space over a field $\mathbf{k}$.

(a) Explain what it means for a list $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ of vectors in V to be (i) **linearly independent**, (ii) **spanning**, and (iii) a **basis**. [**5**]

(b) For each of the following lists of vectors in the real vector space $\mathbb{R}^3$ decide, with justification, whether it is **linearly independent**, whether it is **spanning**, and whether it is a **basis** of the vector space V (you may assume without proof that $\dim(\mathbb{R}) = 3$):

(i) $(1, 0, 1)$, $(0, 5, 0)$, $(0, 0, 0)$;

(ii) $(2, 1, 0)$, $(1, 1, 1)$, $(-1, 1, 2)$, $(0, 4, 1)$;

(iii) $(3, 0, 0)$, $(1, 0, 1)$, $(0, -2, 0)$.

[**7**]

(c) (i) Define the **dimension** of V in terms of bases of V.

(ii) State, without proof, a result relating the length of a spanning list of vectors in V to that of a linearly independent list.

(iii) Using the result in Part (ii), show that the dimension of V is well defined.

[**6**]

(d) Prove the following: A minimal spanning list $B = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of vectors in V is a basis of V. [**5**]

Question 2 [18 marks].

(a) (i) Let U and W be subspaces of the vector space V. Explain what it means that V is the **sum** of U and W (in symbols $V = U + W$), and what it means that V is the **direct sum** of U and W (in symbols $V = U \oplus W$).

(ii) Show that, if $V = U \oplus W$, then each vector of V can be **uniquely** written in the form $\mathbf{v} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

[**8**]

(b) Let X, Y, D be subspaces of the real vector space $V = \mathbb{R}^2$ given by

$$X = \big\{(x, 0) : x \in \mathbb{R}\big\}, \quad Y = \big\{(0, y) : y \in \mathbb{R}\big\}, \quad D = \big\{(x, x) : x \in \mathbb{R}\big\}.$$

Show that

$$V = X \oplus Y = X \oplus D = Y \oplus D.$$

[**10**]

Question 3 [19 marks].

(a) Describe the **elementary row operations** of types 1, 2, and 3 applied to an $(m \times n)$-matrix A. Define the **elementary matrices** associated with these operations, and explain their connection with row operations on A. **[5]**

(b) Which elementary matrix corresponds to the operation of subtracting twice **column** 3 from **column** 1 in a (3×4)-matrix over the field of real numbers? **[2]**

(c) Define the sign $\operatorname{sign}(\pi)$ of a permutation π on the set $[n] := \{1, 2, \dots, n\}$. Compute $\operatorname{sign}(1_{[n]})$ and $\operatorname{sign}(\tau)$, where τ is some transposition. **[3]**

(d) Write out the Leibniz formula for the determinant of an $(n \times n)$-matrix $A = (a_{i,j})_{1 \le i,j \le n}$. Specialise this formula to the case where $n = 3$ and write down $|A|$ as a sum of six terms. **[6]**

(e) Using Laplace expansion along rows, compute the value of the determinant

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 \\ a_7 & a_8 & a_9 & a_{10} \end{vmatrix}.$$

[3]

Question 4 [26 marks].

(a) Let V and W be vector spaces over the field $\mathbf{k}$.

(i) Define what it means for a map $\varphi : V \to W$ to be **linear**. **[2]**

(ii) Which of the following maps on the real vector space $\mathbb{R}^2$ are linear? Please justify your answer.

(A) $\varphi(x, y) = (x + y, y)$,
(B) $\psi(x, y) = (y, x)$,
(C) $\chi(x, y) = (xy, y)$,

[6]

(iii) Define the **kernel** $\ker(\varphi)$ and the **image** $\operatorname{im}(\varphi)$ of a linear map $\varphi : V \to W$. Show that $\ker(\varphi)$ is a subspace of V.

[6]

(b) Let V be an n-dimensional vector space over the field $\mathbf{k}$, and let $B = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be a basis of V. Let $\varphi : V \to \mathbf{k}^n$ be the map sending a vector $\mathbf{v} \in V$ to its coordinate vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, where $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. Show that φ is linear and a bijection. **[12]**

Question 5 [14 marks].

(a) Define what it means for two $n \times n$-matrices over a field $\mathbf{k}$ to be **similar**. What does it mean to say that an $(n \times n)$-matrix is **diagonalisable**? [**4**]

(b) Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$ (considered as a matrix over the real numbers).

Determine the **eigenvalues** of A, and show that A is **diagonalisable** by exhibiting a **diagonalising matrix** P. Write down the **minimal polynomial** of the matrix A, explaining your reasoning. [**10**]

End of Paper.